**Advanced Math – Pre-Calculus**

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Most of the math symbols in this course were made with Math Type® software. Specific fonts must be installed on the user’s computer for the symbols to be read. Use the pdf format of a document when printing. To copy and paste from the Word document, install the fonts on each computer on which the document is to be used. This can be done by downloading the Math Type® for Windows Font from [http://www.dessci.com/en/dl/fonts/default.asp](http://www.dessci.com/en/dl/fonts/default.asp).
Louisiana Comprehensive Curriculum, Revised 2008
Course Introduction

The Louisiana Department of Education issued the Comprehensive Curriculum in 2005. The curriculum has been revised based on teacher feedback, an external review by a team of content experts from outside the state, and input from course writers. As in the first edition, the Louisiana Comprehensive Curriculum, revised 2008 is aligned with state content standards, as defined by Grade-Level Expectations (GLEs), and organized into coherent, time-bound units with sample activities and classroom assessments to guide teaching and learning. The order of the units ensures that all GLEs to be tested are addressed prior to the administration of iLEAP assessments.

District Implementation Guidelines
Local districts are responsible for implementation and monitoring of the Louisiana Comprehensive Curriculum and have been delegated the responsibility to decide if
- units are to be taught in the order presented
- substitutions of equivalent activities are allowed
- GLEs can be adequately addressed using fewer activities than presented
- permitted changes are to be made at the district, school, or teacher level

Districts have been requested to inform teachers of decisions made.

Implementation of Activities in the Classroom
Incorporation of activities into lesson plans is critical to the successful implementation of the Louisiana Comprehensive Curriculum. Lesson plans should be designed to introduce students to one or more of the activities, to provide background information and follow-up, and to prepare students for success in mastering the Grade-Level Expectations associated with the activities. Lesson plans should address individual needs of students and should include processes for re-teaching concepts or skills for students who need additional instruction. Appropriate accommodations must be made for students with disabilities.

New Features
Content Area Literacy Strategies are an integral part of approximately one-third of the activities. Strategy names are italicized. The link (view literacy strategy descriptions) opens a document containing detailed descriptions and examples of the literacy strategies. This document can also be accessed directly at http://www.louisianaschools.net/lde/uploads/11056.doc.

A Materials List is provided for each activity and Blackline Masters (BLMs) are provided to assist in the delivery of activities or to assess student learning. A separate Blackline Master document is provided for each course.

The Access Guide to the Comprehensive Curriculum is an online database of suggested strategies, accommodations, assistive technology, and assessment options that may provide greater access to the curriculum activities. The Access Guide will be piloted during the 2008-2009 school year in Grades 4 and 8, with other grades to be added over time. Click on the Access Guide icon found on the first page of each unit or by going directly to the url http://mconn.doe.state.la.us/accessguide/default.aspx.
Students in an advanced math course are there because they plan to go to college and want to prepare for college math. Many of them will take the ACT while in the advanced math class and, depending on the school they attend, will need to prepare for placement or credit tests in college algebra. The teacher should go to the ACT website www.actstudent.org/testprep/descriptions/mathdescript.html. This site gives a capsule summary of the six content areas that will be covered on the test along with the percent of questions that come from each of the six areas. There is also a practice test that can be used as a diagnostic tool with your class.

The order in which the units in this course are taught should depend on students and their needs. There are several different ways to arrange the units. Many teachers choose to teach Unit Four on Triangle Trigonometry first. Students enrolled in physics will also need this information. Consider consulting with the physics teacher about the math needed for his/her course and when in the school year it is needed. This will help arrange units accordingly. Four of the sixty-four questions on the ACT test deal with trigonometry. The sample problems on the ACT website are covered in Unit 4.

Unit One on Functions should be the prerequisite for Units Two, Three, and Five. The topics and vocabulary in Unit One will be used when studying various functions in those three units. Consider whether your students will be taking placement or credit tests in trigonometry. This requires much memorization on the part of the student many colleges do not allow a calculator on the credit test. Therefore, Units Five and Six are probably best taught early in the second semester so that the material will be fresh in the students’ minds.
Advanced Math – Pre-Calculus

Unit 1: Functions

Time Frame: 3.5 weeks

Unit Description

Functions are a fundamental mathematical concept and as such have been part of the mathematics curriculum for much of the student’s school career. This unit expands the concepts taught in earlier grades and provides a review of essential mathematical skills needed in the course and in future courses. Technology is used to support and extend the concepts to be studied. All students should have access to a graphing calculator.

Student Understandings

Multiple representations of functions – verbal, numeric, graphic, and algebraic – are emphasized. Students learn to move among the different representations of functions, using the representation that best describes the problem at hand. Vocabulary plays an important part in understanding the concepts introduced in this course. Students build on the vocabulary learned in previous courses. They begin the study of individual functions, reviewing the linear and absolute value function and studying the step function and the piecewise defined function.

Guiding Questions

1. Can students identify functions in their various representations?
2. Can students find the domains and ranges of the function?
3. Can students describe local and global behavior?
4. Can students identify even and odd functions?
5. Can students use translations, reflections, and dilations to write rules and graph new functions from parent functions?
6. Can students combine two functions using the operations of addition, subtraction, multiplication, and division?
7. Can students find and use compositions of functions?
8. Can students verify that two functions are inverses of each other?
9. Can students use the graphs of functions to decide whether functions have inverses?
10. Can students interpret inverse functions within the context of a real-life application?
11. Can students rewrite an absolute value function as a piecewise defined function?
12. Can students sketch the graph of a piecewise defined function?
Unit 1 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations A-1-H)</td>
</tr>
<tr>
<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics for the function (A-3-H)</td>
</tr>
<tr>
<td>8.</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)</td>
</tr>
<tr>
<td>10</td>
<td>Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>16.</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>25.</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>27.</td>
<td>Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology. (P-3-H)</td>
</tr>
<tr>
<td>28.</td>
<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
</tr>
<tr>
<td>29.</td>
<td>Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)</td>
</tr>
</tbody>
</table>

**Sample Activities**

Much of what is covered in this unit was introduced to the student in Algebra II, Units 1 and 7. Prior to the beginning of this unit, the teacher should look at those units to get an idea of what was covered and the vocabulary used. Start with a pre-test covering what students learned in Algebra II about functions to determine what the students remember about the concepts. Pre-test Functions BLM can be found in the Blackline Master section. All of the units in this course include a “spiral.” A spiral consists of 10 problems designed to give students an ongoing review of the concepts and skills they have learned. Sample spirals are included in each unit. The one for this unit covers topics from Algebra II. More information on spirals is given under General Assessments at the end of this unit.
Beginning the Glossary

Materials List: index cards 3 x 5 or 5 x 7, What Do You Know about Functions? BLM, pencil

Each unit will have a glossary activity. Two methods will be used to help the students understand the vocabulary for the unit. Begin by having each student complete a self-assessment of his/her knowledge of terms using a modified vocabulary self awareness chart (view literacy strategy descriptions), What Do You Know About This Topic? BLM. Do not give the students definitions or examples at this stage. Ask the students to rate their understanding of each term with a “+” (understand well), a “?” (limited understanding or unsure), or a “–” (don’t know). Over the course of the unit students are to return to the chart, add information or examples, and reevaluate their understanding of the terms or concepts. Students should also make use of a modified form of vocabulary cards (view literacy strategy descriptions). Vocabulary in this course is extremely important. This is an excellent method to use to keep track of the vocabulary not only in this unit but throughout the course. Many of these terms will be used repeatedly throughout the year. One of the advantages vocabulary cards have over a vocabulary list is that, as students add to their cards over the year, the cards can be kept in alphabetical order. This will make it easy to refer to a term or concept when needed. Students should add a vocabulary card as each term is introduced in the unit. Each card should contain the definition, a description of how the term is used in this course, and one or more examples or illustrations of what the term means. Each student should provide his or her own examples. Both sides of the card can be used if necessary. An example of a card on functions is shown below. Specific examples of each of the representations can be written on the back.

<table>
<thead>
<tr>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>The definition: A function is a correspondence or process that assigns a single output to each member of a given set of inputs.</td>
</tr>
</tbody>
</table>

The four representations of a function:

- **Algebraically**: $y = f(x)$ where $f(x)$ is some expression in terms of $x$
- **Graphically**: The graph of a function is obtained by plotting the set of ordered pairs or the symbolic representation.
- **Numerically**: A set of ordered pairs of its inputs and outputs usually shown as a table
- **Verbally**: A description of a function in words usually related to a actual situation
It would be a good idea if the teacher worked with the class to develop the first card. Once the class understands what should be on a card, developing the cards is something that could be done within a group or with a partner. However, it is necessary that the students’ cards be checked for correctness and completion. Items from the vocabulary should be included as part of the ongoing assessments and the written unit tests.

Some of the vocabulary that the students encounter in this course will be new; others will be known by a different name. Be aware that students might think that upon hearing a term used for the first time in this course that they are learning a new concept when it is another name for a concept already mastered. For instance, the domain of a function might have been referred to in previous courses as “inputs”, “independent variables”, the set of abscissas of the ordered pair, or “the set of all x’s.” Call this to the attention of the students when introducing the vocabulary word.

Vocabulary to know for this unit: function, domain, range, independent variable, dependent variable, open intervals, closed intervals, function notation, vertical line test, implied domain, increasing intervals, decreasing intervals, strictly increasing or strictly decreasing, constant, relative minimum, relative maximum, local extrema, even function, odd function, zeros, translations, reflections, dilations, one-to-one, composition, inverse function, horizontal line test, piecewise functions, continuous functions, functions with discontinuities.

Be sure to continue the symbolic notation that was introduced in Algebra II, Unit 1 – Activity 2 as shown below. Either notation shown is acceptable. Coordinating the notation with the textbook is very helpful.

If \( a < b \) are real numbers then
- \( a \leq x \leq b \) can also be written in interval notation as \([a, b]\)
- \( a < x < b \) can be written in interval notation as \((a, b)\)

The symbols \( \infty \) and \( -\infty \) read as “positive infinity” and “negative infinity” and do not represent real numbers. Therefore
- \( x < a \) is written in interval notation as \((-\infty, a)\)
- \( x \leq a \) is written in interval notation as \((-\infty, a]\)
- \( x > a \) is written in interval notation as \((a, \infty)\)
- \( x \geq a \) is written in interval notation as \([a, \infty)\)

Activity 1: Finding Functions in Situations (GLE: 4, 6, 25, 29)


Students need to be familiar with the following vocabulary for this activity: function, domain, range, independent variable, dependent variable, open intervals, closed intervals, function notation, vertical line test, implied domain
Introduction:
The purpose of this activity is to review and expand the concept of function. It is especially important that students work with the multiple representations of functions – verbal, numeric, graphic, and algebraic. Start with an informal characterization of a function such as

A function is a relationship between two quantities such that one quantity is associated with a unique value of the other quantity. The latter quantity, often called \( y \), is said to depend on the former quantity, often denoted \( x \).

Give a familiar example

The amount we pay for gas and the number of gallons bought.

(1) Assign a price for the gasoline. Ask students to name the independent variable and the dependent variable. This is a case in which the cost can be dependent on the number of gallons, or the number of gallons one is able to buy can depend on the amount of money one has to spend.

(2) Set up a table.

<table>
<thead>
<tr>
<th>Number of gallons ( g )</th>
<th>Cost of gasoline ( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(3) Draw a graph setting up a coordinate system, labeling the axes, and choosing appropriate scales.

(4) Write the rule for this particular function.

Follow this with some more verbal descriptions of functions to which students can relate. Give several illustrations asking the students to name the independent and dependent variables as well as to sketch a graph that might be used. Have the students describe some scenarios. With each example, have the students name the independent and dependent variables. Some examples to use to begin the discussion are as follows:

- The population of bacteria and the time it is allowed to grow
- The cost of postage for a package and its weight
- The time it takes for a rock to hit the ground and the height from which it is dropped
- The balance of a mortgage on a house and time allowed for repayment

Once the students grasp the concept, hand out Finding Functions in Situations BLMs. Have the students work in groups to identify the independent variable, the dependent variable, and to sketch a graph of the relationship in each of these statements. Possible answers are given.
Ask students to use the procedure below to solve a number of real-life problems throughout the course. An example to use with the class follows.

Procedure for Developing a Mathematical Model
1. Set up a table using data that appear to be related.
2. Set up a coordinate system, label axes, and choose appropriate scales.
3. Plot data points as ordered pairs on the coordinate system.
4. Sketch a curve that passes through the points.
5. Describe the functional relationship (or an approximation of it) with a symbolic formula.
6. Use the curve and the equation to predict other outcomes.
7. Consider the reasonableness of the results.
8. Consider any limitations of the model.
9. Consider the appropriateness of the scales.

For this activity the models will be linear. It will give students a chance to review the linear function while at the same time to expand their ideas of function. Use the following problem to illustrate the process.

Joanne is starting an aerobic exercise program and has learned from her class that the heart rate increases with the increasing intensity of exercise. Each of the exercises she tries is labeled with its exercise intensity. She checks her pulse rate response immediately after each routine and obtains the following information:
- At rest her pulse rate is 70.
- For the exercise with 0.30 intensity her pulse rate is 106.
- For the exercise with 0.60 intensity her pulse rate is 142.
- For the exercise at .75 intensity her pulse rate is 160.

She wants to know what her heart rate will be when she exercises at 100 percent intensity.

Using the procedure for developing a mathematical model:
1. Set up a table using data that appear to be related.

<table>
<thead>
<tr>
<th>Intensity</th>
<th>0</th>
<th>0.30</th>
<th>0.60</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pulse Rate</td>
<td>70</td>
<td>106</td>
<td>142</td>
<td>160</td>
</tr>
</tbody>
</table>

2. Set up a coordinate system, label axes, and choose appropriate scales.
3. Plot data points as ordered pairs on the coordinate system.
4. Sketch a curve that passes through the points.
   - The pulse rate is dependent on the intensity of the exercise so the Intensity, \(I\), becomes the independent variable and the pulse rate, \(R\), the dependent variable.
   - Discuss appropriate scales for each axis. In order to obtain an accurate graph it is important that the students use graph paper.
5. Describe the functional relationship (or an approximation of it) with a symbolic formula.
• Accurately plotted the points lie on a line. This can be confirmed by using the slope formula. The linear equation \( R = 120I + 70 \) is obtained.

6. Use the curve and the equation to predict other outcomes.
   • At 100% intensity the pulse rate is 190.

7. Consider the reasonableness of the results.

8. Consider any limitations of the model. Students should note the limits placed on the domain and range.

9. Consider the appropriateness of the scales. Was the scaling chosen appropriate so that the finished graph is accurate? Are the points easily visible?

Information for the model is found on the following website:
http://www.teamoregon.com/publications/polsrate.html

At this point, hand out Solving Problems Using Mathematical Modeling BLMs. Let the students work in groups to solve each of the problems.

Activity 2: Functions and their Graphs (GLEs:: 4, 6, 25, 28)

Materials List: Functions and Their Graphs BLM, graphing calculator, pencil

Students should be familiar with the following vocabulary for this activity: increasing intervals, decreasing intervals, constant, relative minimum, relative maximum, local extrema, even function, odd function, zeros, translations.

Students will use equations and graphs to find the domain and range, increasing and decreasing intervals, relative extrema, and the symmetry of functions in this activity. Again this will be a review of what the student learned in Algebra II. Unit 1 Activity 3 and Unit 7 Activities 1, 2, and 7 cover this material in Algebra II. Problems 2, 6, 9, and 10 on the pretest will give an idea of how much students remember. When discussing the graphs of functions be sure to distinguish between a curve or line that has an end point and one whose end behavior is \(+\infty\) or \(-\infty\). This is a good time to mention asymptotes and how they are shown as a dotted line on a graph. The problems below can also be used as a review. Identify the location of a maximum or minimum (the x-coordinate) and its value (the y-coordinate).

1. Find the domain and range of \( f(x) = \sqrt{2 - x} + 3 \). Support the answer with a graphing utility.
   a) \( 2 - x \geq 0 \) the value under the square root must be \( \geq 0 \)
   b) \( x \leq 2 \) solution of the inequality in part (a)
   c) domain is \( \{x: x \leq 2\} \) Note: Use the type of notation that the textbook uses. The interval notation for the domain would be \((-\infty, 2]\)
   d) the range for a square root is \( y \geq 0 \), but there is a vertical translation upward by 3 units. Therefore, the range is \( \{y: y \geq 3\} \). Using interval notation this would be \([3, \infty)\)
The graph using a TI-83+

If the graph goes to the edge of the window then the end-behavior is $+\infty$ or $-\infty$. The fact that the graph stops at $x = 2$ shows that $x < 2$ or $x \leq 2$. A quick check of the table function shows that the answer with this problem is $x \leq 2$.

Use the graph shown above to find the following:

a) the domain and range of $f$

b) On which intervals is $f$ increasing?

c) On which intervals is $f$ decreasing?

d) What are the zeros of $f$?

e) Where is $f(x) > 0$?

f) Where is $f(x) < 0$
Answers: (a) domain is \( \{x|x > -6\} \), range is \( \{y|y > -3\} \)
(b) increasing \((-3, 0)\)
(c) decreasing \((-6, -3)\) and \((0, \infty)\)
(d) zeros are -2 and 2
(e) \(f(x) > 0 \) \((-\infty, -4)\) and \((-2, 2)\)
(f) \(f(x) < 0 \) \((-4, -2)\) and \((2, -\infty)\)

Once the students understand the vocabulary, have them work the problems on the Functions and Their Graphs BLMs.

**Activity 3: Combinations of Functions (GLEs: 4, 10, 16, 28)**

Materials List: Operations on Functions BLM, graphing calculator, pencil

Students need to be familiar with the following vocabulary for this activity: composition of functions.

Students have learned how to combine functions to create new functions in Algebra II. This activity begins with a review of those operations, then continues with problems composing functions, as well as, practice in combining functions using the four representations of functions. Encourage students to support their answers with a graphing utility. For example, if students are asked to find \(f(g(x))\) where \(f(x) = x^2\) and \(g(x) = \sqrt{4 - x^2}\), they should enter \(y_1 = \sqrt{4 - x^2}\) and \(y_2 = (y_1)^2\) The Operations on Functions BLM for this activity could be assigned as homework or done in class as a group exercise. It will depend on how much the students remember from Algebra II.

**Activity 4: Inverse Functions (GLEs: 4, 6, 8, 10, 25, 27, 29)**

Materials List: Inverse Functions BLM, graphing calculator, graph paper, pencil

Students need to be familiar with the following vocabulary for this activity: inverse function, one-to-one, strictly increasing or strictly decreasing, and horizontal line test.

Be sure to stress that \(f^{-1}\) stands for an inverse function. Students confuse \(f^{-1}\) as a command to find the reciprocal of the function instead of its inverse.

Students have learned to find inverse functions using linear and exponential functions. A review of the algebraic steps is left to a spiral review. This activity is written to help students understand why some functions have an inverse that is a function and some do not.

When a single relationship can result in two functions, those two functions are called inverses of each other. Help students to understand that the independent variable in the first function becomes the dependent variable in the second function.
Examples:

**The amount we pay for gas and the number of gallons bought.**
The total cost of the gasoline is dependent on the number of gallons bought. The ordered pair represents (number of gallons, total cost). Now look at it this way: The amount available to spend on gas will determine how many gallons can be bought. This gives the ordered pair (total cost, number of gallons)

**The number of hours worked and the weekly wage paid**
For a person paid by the hour, the amount he earns per week is dependent on the number of hours worked. The ordered pair is (number of hours worked, total wages paid). It can also be said that the number of hours worked is dependent on the total wage earned. Now the ordered pair is (total wages, number of hours worked).

Now look at Activity 2, Finding Functions in Situations BLM, problem # 2. It doesn’t result in two functions.

**The height of a punted football and the number of seconds since it was kicked.**
“The height of the football depends on the number of seconds that have elapsed since it was kicked” (time, height) is a function. “The time that has elapsed depends on the height of the football” is not a function since there are two times when the ball is $h$ feet off the ground.

Illustrate each of the examples above with tables and graphs. Finally, use the statements in Activity 2, Finding Functions in Situations BLM for more examples. Have the students work together to determine if each statement results in two functions. They should find that items 1, 3, and 5 are strictly increasing functions and therefore have an inverse. Item 4 does not.

Help students to see that when the inverse is also a function, the function itself is called a one-to-one function. One-to-one functions are easy to recognize from their graphs. They pass the horizontal line test since they are either increasing or decreasing throughout their domains. (Many textbooks refer to them as strictly increasing or strictly decreasing functions.)

Hand out the Inverse Functions BLMs. This set of problems can be done in class as a group exercise or as homework.

**Activity 5: Piecewise Defined Functions (GLEs: 4, 25)**

Materials List: Piecewise Defined Functions BLM, pencil

Students need to be familiar with the following vocabulary for this activity: piecewise functions, continuous function, a function with discontinuities,
Piecewise defined functions are functions that use different rules for different parts of the domain. The absolute value function is an example of a piecewise defined function.

\[ f(x) = |x| \]

\[ = \begin{cases} 
  x, & \text{for } x \geq 0 \\
  -x, & \text{for } x < 0 
\end{cases} \]

It is also an example of a continuous function. Point out that the linear function is also a continuous function.

There are a number of real-life applications of piecewise functions ranging from the United States income tax to cost of postal service. This activity is designed to teach the student (1) how to sketch the graph of a piecewise function and (2) how to rewrite an absolute value function as a piecewise defined function. Linear and absolute value equations are used as the rules for the problems. However, this is one of those activities that can be repeated throughout the course using the equations of that particular unit as rules for different pieces of the domain. Add one or two piecewise problems to the spirals for that particular unit.

Teaching example:

Graph \( f(x) = \begin{cases} 
  x + 1 \text{ for } x \leq 2 \\
  x - 3 \text{ for } x > 2 
\end{cases} \)

Begin with the portion \( f(x) = x + 1 \) for the piece of the domain \( x \leq 2 \). Evaluate \( f(2) \) and locate on the graph as a filled circle. Since this is a line, choose another number where \( x < 2 \). Locate that point and draw a line segment between them. An arrow can be placed at the point where \( x < 2 \).

Repeat the process for the portion \( f(x) = x - 3 \). Find \( f(2) \) and locate that point on the graph as an open circle. Choose another number where \( x > 2 \) and locate that point. Connect those two points. An arrow is placed at the point where \( x > 2 \). Point out that this is an example of a discontinuous function with the point of discontinuity being at \( x = 2 \). The graph “jumps” from the point (2, 3) to an open circle at (2, -1). The discontinuity in this problem is labeled a “jump discontinuity.”

Follow this with more examples if necessary. Hand out Piecewise Defined Functions BLM. This is an excellent activity for group work.

**Activity 6: Beginning the Portfolio - Library of Functions (GLEs: 4, 6, 16, 27, 28, 29)**

Materials List: a folder, Library of Functions –Linear Functions BLM, pencil, paper, graph paper

Students will have several entries in the Library of Functions over a period of several months. Each teacher should have a place where the work can be kept. Each of the functions studied in this course will have an entry in the Library of Functions. The
purpose of this ongoing activity is to prepare a portfolio of functions. It is important that
a student have a visual picture of the function, its important characteristics, and what role
it might play in real life. The linear function studied in previous courses should be
completed in this unit. The student should begin with the parent function – for linear
functions \( f(x) = x \). In many books this is referred to as the Identity Function. In general
the following characteristics should be covered:

- domain and range
- local and global characteristics such as symmetry, continuity, whether the
  function has local maxima and minima with increasing/decreasing intervals,
  or a strictly increasing or strictly decreasing function with existence of an
  inverse, end-behavior
- discuss the common characteristics of linear functions such as the constant
  rate of change, the existence of a y-intercept, and a zero in all functions in
  which the slope \( \neq 0 \).
- examples of translation, reflection, and dilation in the coordinate plane
- a real-life example of how the function family can be used showing the 4
  representations of a function along with a table of select values
- describing the slope, y-intercept, and zeros within the context of the example
  used by the student

Sample Assessments

General Assessments

- The student will perform a writing assessment which covers activities of the unit
  and which uses the glossary created throughout this unit. Look for understanding
  of concepts are used, by using directions such as explain, justify, or compare and
  contrast. Some possible topics are given below:
  1. Explain to a friend how to find the domain and range of a function.
  2. What is the inverse of a function? Is the inverse always a function? How are
     the domains and ranges of functions and their inverses related? Given the
     graph of a function how do you tell if it has an inverse?
  3. Given \( f(x) = x^2 \) and \( g(x) = 1 - 2x \), find \( g(f(2)) \). Explain how to work this
     problem.
  4. What is an even function, an odd function? What symmetries do the graphs of
     such functions have? Give examples. Give an example of a function that is
     neither even nor odd.
- The student will review previously learned concepts throughout the unit. One
  favorite method is a weekly “spiral”, a handout of 10 or so problems covering
  work previously taught in the course. Spirals can also be study guides for a unit
  test or a midterm exam. The General Assessments Spiral BLM covers some of
  the topics learned in Algebra II. Design other spirals that review what students
  missed on the pretest as well as using problems that (1) reinforce the concepts
learned in earlier activities, (2) review material taught in earlier courses, or (3) review for the ACT or SAT.

- The students will turn in this entry for the Library of Functions BLM for an assessment. Student instructions for the entry can be found in the blackline masters for Unit 1.

**Activity-Specific Assessments**

- **Activity 1:** The students will engage in a group activity using problems similar to those used in Activity 1 and worked in the Solving Problems Using Mathematical Modeling BLM. The student will use the Procedure for Developing a Mathematical Model and will be evaluated using a rubric based on
  1. teacher observation of group interaction and student work
  2. work handed in by each member of the group
  3. an explanation of each group’s problem to the class
One method of determining a group grade is to randomly select a paper from each group and to grade that paper. The grade for that paper will provide the group grade for the hand-in portion of the grade. If there is time to have each group explain a particular problem, then the choice of the person to explain the problem for the group is by random selection.

- **Activity 3:** The student will demonstrate proficiency in working problems covered throughout this unit. Choice of those problems will depend on classroom progress as well as previous knowledge of the material. Below is a set of problems that could go with the material covered in Activity 4.

  Given:
  \[ f(x) = x^2 - x - 6 \text{ and } g(x) = \frac{1}{x - 1} \]

  Find:
  a) \((f + g)(x)\)
  b) \((fg)(x)\)
  c) \(\left(\frac{f}{g}\right)(x)\)
  d) \(f(g(x))\)

- **Activity 5:** The students will demonstrate proficiency (a) in writing absolute value functions as piecewise defined functions and (b) in graphing piecewise functions. Problems should include ones such as shown in the examples below:

  1) Graph \(f(x) = 2x - 3,\) \(x \leq 1\)
     \[2 - x, \quad x > 1\]

  2) Rewrite as a piecewise function and graph:
     \(f(x) = |4x - 2|\)
Advanced Math – Pre-Calculus
Unit 2: Polynomial and Rational Functions

Time Frame: 4.5 weeks

Unit Description

This unit focuses on the polynomial and rational functions. Emphasis is placed upon analyzing each function and using that information to sketch the graphs. Applications using the quadratic and cubic polynomials and rational functions are part of the unit.

Student Understandings

Students recognize the algebraic and graphical representations of polynomial and rational functions. They can identify the local and global characteristics of those functions and can use their knowledge to sketch a graph of the function and to obtain a complete graph using a graphing utility. They are able to use both types of functions to model and solve real-life problems.

Guiding Questions

1. Can students identify polynomials given the equation or a graph?
2. Can students sketch and analyze graphs of polynomial functions using the zeros, local maxima and minima, and end-behavior?
3. Can students determine the roots of a polynomial equation and apply the fundamental theorems of algebra?
4. Can students use quadratic and cubic functions to model real-life problems?
5. Can students identify rational functions given the equation or a graph?
6. Can students determine the discontinuities of a rational function and name its type?
7. Can students find horizontal and vertical asymptotes of rational functions?
8. Can students analyze and sketch graphs of rational functions?
9. Can students model and solve real-life problems using rational functions?
## Unit 2 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
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<tbody>
<tr>
<td><strong>Number and Number Relations</strong></td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>Read, write, and perform basic operations on complex numbers. (N-1-H) (N-5-H)</td>
</tr>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics for the function (A-3-H)</td>
</tr>
<tr>
<td>7.</td>
<td>Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions. (A-3-H)</td>
</tr>
<tr>
<td>8.</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)</td>
</tr>
<tr>
<td>9.</td>
<td>Solve quadratic equations by factoring, completing the square, using the quadratic formula, and graphing (A-4-H)</td>
</tr>
<tr>
<td>10.</td>
<td>Model and solve problems involving quadratic, polynomial, exponential logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>16.</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors and matrices (G-3-H)</td>
</tr>
<tr>
<td><strong>Data Analysis, Probability, and Discrete Math</strong></td>
<td></td>
</tr>
<tr>
<td>19.</td>
<td>Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>24.</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>25.</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>27.</td>
<td>Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H)</td>
</tr>
<tr>
<td>28.</td>
<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
</tr>
<tr>
<td>29.</td>
<td>Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)</td>
</tr>
</tbody>
</table>
Sample Activities

Students were introduced to polynomials and rational functions in Algebra II, Units 2, 3 and 5. Before beginning this unit, look at those units to get an idea of what was covered and the vocabulary used. Begin this unit by giving the students Unit 2, Pretest on Polynomials and Rational Functions BLMs. This will give a good idea of how much the students remember. Spirals throughout this unit should reinforce those concepts in which the students are weak. Calculators should not be used on this pretest.

Ongoing: Glossary

Materials List: index cards, What do You Know About Polynomials and Rational Functions BLM, pencil

Have students continue the glossary activity in this unit. Repeat the two methods used in unit 1 to help the students understand the vocabulary for unit 2. Begin by having each student complete a self-assessment of his/her knowledge of the terms using a modified vocabulary self awareness chart (view literacy strategy descriptions) What Do You Know About Polynomials and Rational Functions BLM. Do not give the students definitions or examples at this stage. Ask the students to rate their understanding of each term with a “+” (understand well), a “?“ (limited understanding or unsure) or a “–“ (don’t know). Over the course of the unit, students are to return to the chart, add information or examples, and re-evaluate their understanding of the terms or concepts. Students should continue to add to the vocabulary cards (view literacy strategy descriptions) introduced in Unit 1. Make sure that the students are staying current with the vocabulary cards. Time should be given at the beginning of each activity for students to bring their cards up to date.

Terms to add to the vocabulary list: polynomial, terms, factors, leading coefficient, zeros, multiplicity of zeros, the connection between zeros of a function, roots of an equation, and x-intercepts of a graph; the difference between f(x) = 0 and f(0), quadratic, cubic, parabola, axis of symmetry, vertex, continuous, local extrema, leading coefficient, intermediate value theorem, synthetic division, remainder theorem, factor theorem, rational root theorem, rational functions, asymptotic discontinuity, vertical asymptote, horizontal asymptote, concavity

Activity 1: A Review of the Quadratic Function (GLEs: 4, 6, 7, 9, 10, 27)

Materials List: A Review of the Quadratic Function BLM, graphing calculator, paper, pencil

Students need to be familiar with the following vocabulary for this activity: terms, factors, multiplicity of zeros, the connection between zeros of a function, roots of an
equation, and $x$-intercepts of a graph; the difference between $f(x) = 0$ and $f(0)$, quadratic, parabola, axis of symmetry, vertex, concavity.

Quadratic functions are introduced and studied extensively in Algebra II. The pretest should give some help in determining how much of the material students have retained. It may be that A Review of the Quadratic Function BLM will suffice. If not, spiral reviews on the topic will be needed. This activity gives students the opportunity to review the use of the discriminant in determining the nature of the zeros, finding the zeros, writing equations given the graphs of a quadratic, translating between the symbolic and graphical forms of a function, and applying quadratics in real-life situations. Question #9 asks that students add the Quadratic Function to their Library of Functions. It is also part of the blackline master for Activity 8. The teacher has a choice of when this should be assigned.

Hand out the Review of the Quadratic Function BLMs. Let students work together on this activity.

**Activity 2: Discovery using Technology (GLEs: 6, 7, 8, 19)**

Materials List: Discovery Using Technology BLM, graphing calculator, paper, pencil

Students need to be familiar with the following vocabulary for this activity: polynomial, linear, cubic, leading coefficient.

The ability to recognize a polynomial function from either the equation or the graph, to identify an even or odd degree polynomial, to know if the leading coefficient is positive or negative, and to know something about the general shape of the graph for small values of $x$ is one of the basic student understandings for this unit. It is also very important that the student understands that a polynomial is a continuous function whose graph has smooth curves.

While a graphing utility is certainly a help in obtaining a picture of a polynomial function, students often have a problem in setting the window in order to obtain a complete graph. This activity should help the student to analyze the polynomial function for large and small values of $x$ so that a complete graph can be obtained using a graphing utility. The zoom out feature on the graphing utility will help the students obtain the answer for large values of $x$, while the zoom in feature will help with the small values of $x$. The table function also helps the student to see what is happening. For the large values of $x$, have the students set the table at $x = 10$ and set $\Delta \text{Tbl}$ to 10. For small values of $x$, have the student set the table at $x = -.5$ and set $\Delta \text{Tbl}$ to 0.1. If the students have not had the experience of using the table function, demonstrate these activities with a polynomial such as $P(x) = x^3 - 2x^2 - 3x + 4$ before letting students start the activity. To demonstrate the table function set the table at $x = -10$ and $\Delta \text{Tbl}$ to 1.
This activity is also a writing exercise. Be sure the students answer each question in complete sentences – a word or two does not suffice. Hand out the Discovery Using Technology BLMs letting students work together to gather their information. However, the written answers to the questions should be done alone and handed in for a grade.

Activity 3: The Zeros of Polynomials (GLEs: 1, 4, 6, 9)

Materials List: The Zeros of Polynomials BLM, pencil

Students need to be familiar with the following vocabulary for this activity: intermediate Value Theorem, synthetic division, Remainder Theorem, Factor Theorem, Rational Root Theorem.

The Zeros of Polynomials BLM is a non-calculator exercise designed to give students practice in finding the zeros of a polynomial using synthetic division, the factor theorem, the rational root theorem and the quadratic formula where necessary. This was taught in Algebra II. There were also some problems on this topic on the pretest. How the students performed on the pretest and how well they answer the questions during the introductory remarks will give an idea of whether this material should be treated as a review or as new material. It may be necessary to refer to the textbook for additional problems should students feel uncomfortable with the concepts. Students should be able to find zeros of a cubic function both with and without a calculator. The Zeros of Polynomials BLM can be used either for class work or homework depending on student proficiency.

Activity 4: Analyzing Polynomials (GLEs: 1, 4, 6, 7, 8)

Materials List: Analyzing Polynomials BLM, graphing calculator, graph paper, pencil

In the previous exercises the students have learned to identify the end-behavior of a polynomial function and find the zeros. In Unit 1, they identified increasing and decreasing intervals and the relative extrema from a graph. Without a graphing utility, students would need calculus to do the same with the polynomials of degree greater than two. However with technology the relative extrema can be identified and the increasing and decreasing intervals found. In this activity, students will analyze each of the polynomials. Hand out the Analyzing Polynomials BLMs. Students will be asked to find the local and global characteristics of each polynomial, and then graph each one using the information they found.
Activity 5: Applications of Polynomial Functions (GLEs: 4, 6, 8, 10, 24, 25, 27, 29)

Materials List: Applications of Polynomial Functions BLM, graphing calculator, pencil, paper

In this activity students have to translate a verbal description into an algebraic model. Students might find the strategy outlined below helpful in solving the problems.

✓ Step 1: Draw a diagram if needed.
✓ Step 2: Identify the dependent variable.
✓ Step 3: Identify the constraints.
✓ Step 4: Express the dependent variable as a function of a single independent variable. Identify the domain of the independent variable.
✓ Step 5: Graph the independent variable as a function of the independent variable.
✓ Step 6: Identify the locations and values of the maximum or minimum.

Example problem to use with the class:

Mr. Johnson is going to build a set of dog runs for his two retrievers. He plans to enclose a rectangular area and divide it down the middle. He has 120 feet of fencing to use. What dimensions should he use to maximize the area?

Step 1: Draw the diagram. Let the class decide what variables to use and where to put them.

Step 2: Identify the dependent variable. Area is the dependent variable. It depends on the length and width chosen for the rectangular area. We can write \( xy = A \)

Step 3: Identify the constraints. There is only 120 feet of fencing to use around the perimeter and to divide the two runs. This gives us \( 3x + 2y = 120 \) as the constraint on the problem.

Step 4: Express the dependent variable as a function of a single independent variable. Identify the domain of the independent variable. Solve the constraint equation for \( y \) and substitute it into the area equation obtaining:

\[ A = x(60 - 1.5x) \]

Area must be positive so the domain must be \( 0 < x < 40 \) for this problem.

Step 5: Graph it! Set \( xmin = 0 \) and \( xmax = 40 \).
Step 6: Identify the locations and values of the maximum.

Mr. Johnson will cut each of the x-lengths at 17 feet. This will leave him with 34.5 feet for each of the 2 y-lengths. The area of the two runs together will be 586.5 square feet. Each run will have the area of 293.25 square feet.

Hand out the Applications of Polynomial Functions BLMs and let the students work in groups to solve each problem.

Activity 6: Rational Functions and Their Graphs (GLEs: 4, 6, 7, 8, 19, 27)

Materials List: Rational Functions and their Graphs BLM, graph paper, graphing calculator, pencil

Students need to be familiar with the following vocabulary for this activity: rational functions, asymptotic discontinuity, vertical asymptote, horizontal asymptote.
Before beginning this activity students should have learned
✓ how to determine the domain of the function
✓ how to determine the type of discontinuity (vertical asymptotes and “holes” in the graph) that result from those x-values not in the domain of the function
✓ how to find zeros of the function
✓ how to find horizontal asymptotes by examining the function as
\[ x \to \infty \text{ and } x \to -\infty \]

Work one or two example problems for the class prior to their working the Rational Functions and Their Graphs BLM.

\[ 1. f(x) = \frac{x-2}{x+3} \quad \text{and} \quad 2. f(x) = \frac{x+2}{x^2-16} \]

Use the step by step method found on the blackline master.

Example 1:

a) The domain is \( \{x: x \neq -3\} \). It is found by setting the denominator \( = 0 \).

b) The zero is 2. It is found by setting the numerator \( = 0 \). Note: remind the students that they should always find the domain first. There may be times that the numerator has the same value. Then that value is not a zero but the x-coordinate of a point that is a “hole” in the graph.
c) -3 is not in the domain so \( x = -3 \) is a vertical asymptote. Be sure that the students understand that they are writing the equation of a vertical line. Just answering 3 would be incorrect. \( y = 1 \) is the horizontal asymptote.

e) The \( y \)-intercept is \(-\frac{2}{3}\).

f) Students should put the vertical and horizontal asymptotes on the graph paper as dotted lines. They should locate the zero at \( x = 2 \) and the \( y \)-intercept at \( y = -\frac{2}{3} \). Find \( f(-4) \) and \( f(3) \) and plot those points. The two parts of the graph should curve following the asymptotes \( x = -3 \) and \( y = 1 \).

Example 2:

a) The domain is \( \{x: x \neq -4 \text{ and } 4\} \). In problem #2 there will be two vertical asymptotes, \( x = -4 \) and \( x = 4 \).

b) the zero is -2.

c) the \( y \)-intercept is \(-\frac{1}{8}\).

Plot the zero and the \( y \)-intercept. Draw a smooth curve through those points that follows the vertical asymptotes.

d) What values does \( y \) take on when \( x < -4? \ x > 4? \) This tells whether the graph is above or below the \( x \)-axis. Remind the students that there are only 2 values missing from the domain. When \( x < -4 \), \( y < 0 \) and when \( x > 4 \), \( y > 0 \). Therefore, the graph when \( x < -4 \) is in quadrant III and following the asymptotes; when \( x > 4 \) the graph is in quadrant 1 and following the asymptotes. Choose two or more values for \( x < -4 \) and \( x > 4 \). Plot those points. Draw a smooth curve that follows the horizontal and vertical axis.

Students should be encouraged to graph each of the rational functions by hand then to check their answers with a graphing calculator. Remind them that when they graph using the calculator, the vertical line graphed is not part of the graph. It represents the vertical asymptote. Once all questions have been answered, hand out the Rational Functions and Their Graphs BLMs. Let the students work either with partners or in a group.

Activity 7: Playing Mr. Professor (GLEs: 6, 27)

Materials List: equipment to write on board, graph paper, pencils

Students will gain some additional practice in graphing rational functions.

Introduce the professor know-it-all strategy (view literacy strategy descriptions). Divide the students into groups. Explain to the students that each group will be called upon to become a team of math prodigies. They will have a chance to demonstrate their expertise.
in graphing rational functions by answering the questions posed by the rest of the class. The team may confer on each question, but each member of the group should have a chance to explain. The questions should cover the points below.

✓ domain
✓ type of discontinuities
✓ vertical asymptotes
✓ horizontal asymptotes
✓ zeros
✓ y-intercept
✓ symmetry

Once all of the points have been covered, the graph of the function should be sketched using the answers given by the group. The other students may challenge any of the answers given. Use one of the problems from Activity 7 to demonstrate to the students how the group should respond to their peers’ questions. Some functions to use are:

\[ f(x) = \frac{x^2 - 1}{x^2 - 4x - 5} \quad 2. \quad f(x) = \frac{x + 3}{x - 5} \quad 3. \quad f(x) = \frac{1}{x^2 - 1}, \]

\[ 4. \quad f(x) = \frac{x}{x^2 - 1} \quad 5. \quad f(x) = \frac{1}{x^2 + 1} \]

Answers for the problems above:

Problem # 1

- **Domain** \( x: x \neq -1, 5 \)
- **Zero at** \( x = 1 \)
- **There is a hole at** \( x = -1 \) **and a vertical asymptote** \( x = 5 \)
- **Horizontal asymptote is** \( y = 1 \)
- **y-intercept is** \( 1/5 \)
- **The hole in the graph is** \( -1, 1/3 \)
- **\( f(6) = 5 \)**
- **no symmetry**

The graph as shown on calculator:

Problem # 2

- **Domain** \( x: x \neq 5 \)
- **Zero at** \( x = -3 \)
- **There is a vertical asymptote** \( x = 5 \)
- **Horizontal asymptote is** \( y = 1 \)
• y-intercept is \(-3/5\)
• \(f(2) = -1.667\)
• \(f(6) = 9\)
• no symmetry

The graph as shown on calculator:

Problem # 3
• Domain \(\{x: x \neq -1, 1\}\)
• Zero: none
• There is a vertical asymptote \(x = -1, x = 1\)
• Horizontal asymptote is \(y = 0\)
• y-intercept is 1
• \(f(-2) = 1/3\)
• \(f(2) = 1/3\)
• this is an even function symmetrical about the y-axis \(f(x) = f(-x)\)

The graph as shown on calculator:

Problem #4
• Domain \(\{x: x \neq -1, 1\}\)
• Zero: 0
• Vertical asymptotes at \(x = 1\) and \(x = -1\)
• Horizontal asymptote is \(y = 0\)
• y-intercept is 0
• \(f(-2) = -2/3\)
• \(f(2) = 2/3\)
• this is an odd function; symmetric with respect to the origin \(f(-x) = -f(x)\)
The graph as shown on calculator:

Problem #5

- Domain \( \{x: x \text{ is the set of reals}\} \)
- Zero: none
- There are no vertical asymptotes
- Horizontal asymptote is \( y = 0 \)
- \( y \)-intercept is 1
- \( f(-1) = 1/2 \)
- \( f(1) = 1/2 \)
- this is an even function

The graph as shown on calculator:

Activity 8: Adding to the Portfolio - Library of Functions (GLE 4, 6, 16, 27, 28, 29)

Materials List: Library of Functions – Quadratic Functions, Polynomial Functions and Rational Functions BLM, paper, pencil

Polynomial and rational functions should be added to the Library of Functions at this time. For each function, students should present the function in each of the 4 representations. They should consider

- domain and range
- local and global characteristics such as continuity, concavity, symmetry, local maxima and minima, increasing/decreasing intervals, and zeros
- examples of translation in the coordinate plane
- possibilities of inverse functions
- a real-life example of how the function can be used
Hand out the Library of Functions – Quadratic Functions, Polynomial Functions and Rational Functions BLM to each student.

**Sample Assessments**

**General assessments**

- The students will perform a writing assessment. They have added to their notebook glossary throughout this unit. They have also had a short writing assignment with each of their activities. Therefore, one of the assessments should cover this material. Look for understanding in how the term or concept is used. Use verbs such as show, describe, justify, or compare and contrast. Some possible topics include:

  **Polynomials**
  a) What is a zero of a function?
  b) What do we mean by a double zero?
  c) Describe the end-behavior of a polynomial of even degree.

  **Rational functions**
  a) How is the zero of a rational function found?
  b) A rational function has a zero between two vertical asymptotes. What is that portion of the graph going to look like?
  c) How are the discontinuities of a rational function related to the domain of a rational function?
  d) What do we mean by the “end-behavior” of a rational function?

- Review of previously learned concepts should be ongoing throughout the unit. Continue a weekly “spiral”, a handout of 10 or so problems covering work previously taught in the course. General Assessments Spiral BLM should be done after the students have reviewed quadratic functions. Many of the concepts introduced in Unit 1 can be reviewed in this unit using the polynomial and rational functions. For instance, give students a graph of a polynomial and ask them to identify the local maxima and minima and the intervals where the function increases and decreases. Another possible problem would be to find the inverse of a rational function such as

  \[ f(x) = \frac{x + 1}{x - 3} \]. Its inverse is \( f^{-1}(x) = \frac{-3x - 1}{1 - x} \). Students can then find the domain and range of both functions since the domain of \( f \) is the range of \( f^{-1} \) and the range of \( f \) is the domain of \( f^{-1} \). Both functions when graphed would be symmetrical about \( y = x \). This is also a good opportunity to have students use interval notation. Have them write \([2, 5]\) instead of \(2 \leq x \leq 5\) and \((2, 5)\) instead of \(2 < x < 5\). Students continue to need practice in working with function notation. Questions such as
Given \( f(x) = 2x^2 - 3x + 4 \), find (a) \( f(3) \), (b) \( f(a) \), or (c) \( \frac{f(x+h) - f(x)}{h} \) will give students such practice.

- Students should turn in the entries for the Library of Functions BLMs for an assessment.
  Scoring rubric should include the following:
  1. Thorough coverage of the material.
  2. The material presented is accurate.
  3. The work is neat and organized. Descriptions are given in sentences.
  4. The graphs are labeled, drawn to scale and are correct.

### Activity-Specific Assessments

- **Activities 2 and 3:** Students should demonstrate proficiency in working problems such as the one below:

  Graph the function \( y = 3x + 2x^2 - x^3 \).
  1. What are the zeros of this function?
  2. How do the zeros relate to the factors of this function?
  3. Which term do you think determines the shape for very large positive and negative values of \( x \)?
  4. Which term do you think determines the shape for very small positive and negative values of \( x \)?

- **Activities 5 or 6:** The students should work with a group on problems similar those in these activities.
  Scoring rubric based on
  1. teacher observation of group interaction and work
  2. explanation of each group’s problem to class
  3. work handed in by each member of the group

- **Activity 7:** Students should be able to graph a rational function with and without a calculator.
Advanced Math – Pre-Calculus
Unit 3: Exponential and Logarithmic Functions

Time Frame: 4.5 weeks

Unit Description

This unit focuses on the exponential and logarithmic functions using the four representations of functions. This unit expands the concepts taught in earlier courses, providing a review of essential mathematical skills and algebraic concepts needed in this course and in future courses. Modeling real-life problems using exponential growth and decay, as well as fitting exponential and logarithmic models to sets of data, play a large part in the unit.

Student Understandings

Students recognize, evaluate, and graph exponential and logarithmic functions. The laws of exponents and logarithms are reviewed, and then used to evaluate, simplify expressions and solve equations. They are able to use both functions to model and solve real-life problems.

Guiding Questions

1. Can students recognize exponential functions in each of the function representations?
2. Can students identify the growth or decay factor in each of the exponential functions?
3. Can students graph exponential functions?
4. Can students recognize, evaluate, and graph exponential functions with base e?
5. Can students use exponential functions to model and solve real-life problems?
6. Can students recognize and graph logarithmic functions with any base?
7. Can students use logarithmic functions to model and solve real-life problems?
8. Can students use the properties of exponents and logarithms to simplify expressions and solve equations?
9. Can students rewrite logarithmic functions with different bases?
10. Can students use exponential growth and decay functions to model and solve real-life problems?
11. Can students fit exponential and logarithmic models to sets of data?
## Unit 3 Grade-Level Expectations (GLEs)

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<tr>
<td>2</td>
<td>Evaluate and perform basic operations on expressions containing rational exponents (N-2-H)</td>
</tr>
<tr>
<td>3</td>
<td>Describe the relationship between exponential and logarithmic equations (N-2-H)</td>
</tr>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>6</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics for the function (A-3-H)</td>
</tr>
<tr>
<td>7</td>
<td>Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)</td>
</tr>
<tr>
<td>8</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)</td>
</tr>
<tr>
<td>10</td>
<td>Model and solve problems involving quadratic, polynomial, exponential logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
<tr>
<td><strong>Data Analysis, Probability, and Discrete Math</strong></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>25</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>27</td>
<td>Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H)</td>
</tr>
<tr>
<td>28</td>
<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
</tr>
<tr>
<td>29</td>
<td>Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)</td>
</tr>
</tbody>
</table>
Sample Activities

Start with the idea that students have been exposed to the properties of exponents and logarithms; that they understand that $y = b^x$ and $\log_b y = x$ are equivalent expressions; that they are able to work with rational exponents; and that they recognize the graphs of the exponential and logarithmic functions. Students studied exponents in both Algebra I and Algebra II. Unit 6 in Algebra II is devoted to exponential and logarithmic functions. Before beginning this unit, the teacher should be familiar with what was covered and the vocabulary used in those Algebra II units. Begin this unit by giving the students the Pretest BLM. This will give a good idea of how much the student remembers. Spirals throughout this unit should reinforce those concepts in which the students are weak.

Ongoing: The Glossary

Materials List: index cards 3 x 5 or 5 x 7, What Do You Know About Exponential and Logarithmic Functions? BLM, pencil

The two methods used to help the students understand the vocabulary for this course. As was done in Units 1 and 2, begin by having each student complete a self-assessment of his/her knowledge of the terms for this unit using the modified vocabulary self awareness (view literacy strategy descriptions) What Do You Know About Exponential and Logarithmic Functions? BLM. Students should continue to make use of a modified form of vocabulary cards (view literacy strategy descriptions). Add new cards for the following terms as they are encountered in the unit: algebraic functions, transcendental functions, exponential functions, symbolic representation of an exponential function, exponential growth model, exponential decay model, logarithmic functions, relationship of exponential and logarithmic functions, natural base e, growth factor, growth rate, exponential growth model, exponential decay model, natural logarithm, common logarithm, logarithm to base a, change-of-base formula, properties of exponents, laws of logarithms, rational exponents, compound interest, half life.

Activity 1: The Four Representations of Exponential Functions (GLEs: 4, 7, 8, 10, 19, 29)

Materials List: The Four Representations of Exponential Functions BLM, calculator, graph paper, pencil

Students need to be familiar with the following vocabulary for this activity: algebraic functions, transcendental functions, growth factor, growth rate, exponential functions, symbolic form of an exponential function, exponential growth model, and exponential decay model.
Unit 2 dealt with polynomial and rational functions both of which are examples of algebraic functions. Algebraic functions are functions whose symbolic form deals with the algebraic operations of addition, subtraction, multiplication, division, raising to a given power, and extracting a given root. The exponential and logarithmic functions studied in this unit are examples of transcendental functions.

In general, exponential models arise whenever quantities grow or shrink by a constant factor, such as in radioactive decay or population growth. Some of the problems in this activity will require the students to determine whether or not the data is exponential by looking for this constant factor. Two examples are shown below.

Example 1: One hundred dollars is invested in a savings account earning 5% per year as shown below:

<table>
<thead>
<tr>
<th>Year</th>
<th>Money in Bank</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$100.</td>
</tr>
<tr>
<td>2</td>
<td>$105.</td>
</tr>
<tr>
<td>3</td>
<td>$110.25</td>
</tr>
<tr>
<td>4</td>
<td>$115.76</td>
</tr>
<tr>
<td>5</td>
<td>$121.55</td>
</tr>
</tbody>
</table>

Is the growth of the money exponential? If it is exponential, what is the growth factor? To determine the answers, look at the ratios of the successive values of $M$ (money in the bank):
\[
\frac{110.25}{105.25} = \frac{115}{110.25} = \frac{121.55}{121.55}.
\]
Each of these ratios equals 1.05. Therefore the growth of money is exponential and the constant growth factor is 1.05.

Example 2: Musical Pitch

The pitch of a musical note is determined by the frequency of the vibration which causes it. The A above middle C on the piano, for example, corresponds to a vibration of 440 hertz (cycles per second). Below is a table showing the pitch of notes above that A.

<table>
<thead>
<tr>
<th>Number $n$, of octaves above this A</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of hertz, $V = f(n)$</td>
<td>440</td>
<td>880</td>
<td>1760</td>
<td>3520</td>
<td>7040</td>
</tr>
</tbody>
</table>

Set up ratios of successive values of $V$:
\[
\frac{880}{440}, \frac{1760}{880}, \frac{3520}{1760}, \frac{7040}{3520}
\]
Each of the ratios is equal to 2, so the function is exponential and 2 is the growth factor. This leads to the algebraic representation of an exponential growth/decay function:
\[ P = P_o a^t, \ a > 0, \ a \neq 1 \]
where $P_o$ is the initial quantity and the vertical intercept of the graph, $a$ is the base or growth/decay factor, $t$ is the time involved, and $P$ is the quantity at time $t$. Some textbooks will use $A_o$ and $A$ instead of $P_o$ and $P$. The teacher may want to change the BLM for this activity to reflect the textbook notation.
Exponential decay occurs when the growth factor is less than 1. If $r$ is the growth rate, then $1 + r$ is the growth factor and $1 - r$ is the decay factor. For instance, the problem may read that the depreciation of a car value is 12%. This means that we have a decay factor with .12 being the value of $r$ so $1 - .12 = .88$. The decay factor is .88. The decay rate is .12 or 12%. The equation would then be $P = P_0 (.88)^t$.

Once the students grasp these concepts, hand out The Four Representations of Exponential Functions BLMs. This is a good classroom activity either for partners or for group work.

The students will be asked to graph the set of values in #2. Be sure that they use graph paper and the appropriate scaling. They should be able to answer the questions using the table and graph. The given equation should only be used to check the answer.

**Activity 2: Continuous Growth and the number $e$ (GLEs: 7, 10, 24, 25)**

Materials List: Continuous Growth and the Number $e$ BLM, calculator, pencil, paper

Vocabulary to be covered for this activity: natural base $e$, compound interest

Prior to this activity, students should be reintroduced to the irrational number $e$ and to the function $f(x) = e^x$. It helps to have the students graph the three functions $y_1 = 2^x, y_2 = e^x, y_3 = 3^x$ to see the place of $e$ on the number line. The graphs on the TI-83 below use a window of Xmin .5, Xmax 1.5, Ymin -1, and Ymax 4. Pressing trace will show the values of $2^1$, $e^1$, and $3^1$.

Once this is done the idea of continuous growth can be introduced. Thus far students have been working with the exponential function $P = P_o a^t$ where $P_o$ represents the initial amount, $a$ the growth factor, and $t$ the amount of time that has elapsed. If the growth is continuous then $a$ is equal to $e^k$ for some $k$. If $a > 1$ (exponential growth), then $k > 0$. If $a < 1$ (exponential decay), then $k < 0$. The equation for continuous growth can be written as $P = P_o e^{k^t}$, $P$ is growing or decaying at a continuous rate of $r$. 

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Example:
One of the most familiar examples of continuous growth is that of compound interest. Money can be invested at an annual rate of interest or it can be compounded quarterly (four times a year), monthly, daily, or continuously. Let $n$ be the number of times a year the initial amount is compounded. This would give us the following formula

$$P = P_0 \left(1 + \frac{r}{n}\right)^{nt}$$

where $n$ is the number of compounding periods. What if the money was compounded continuously? In general, if you invest $P_0$ dollars at an annual rate $r$ (expressed as a decimal) compounded continuously, then $t$ years later your money would be worth $P_0 e^{rt}$ dollars.

Suppose $100$ is invested for 5 years at a rate of $5\%$. How much would we have using each of the compounding periods?

<table>
<thead>
<tr>
<th>number of periods</th>
<th>yearly</th>
<th>quarterly</th>
<th>monthly</th>
<th>daily</th>
<th>continuously</th>
</tr>
</thead>
<tbody>
<tr>
<td>formula</td>
<td>$1000 \left(1.05^\frac{5}{4}\right)$</td>
<td>$1000 \left(1 + \frac{0.05}{4}\right)^{4 \cdot 5}$</td>
<td>$1000 \left(1 + \frac{0.05}{12}\right)^{12 \cdot 5}$</td>
<td>$1000 \left(1 + \frac{0.05}{365}\right)^{365 \cdot 5}$</td>
<td>$1000e^{0.05 \cdot 5}$</td>
</tr>
<tr>
<td>amt saved</td>
<td>$1276.28$</td>
<td>$1282.04$</td>
<td>$1283.36$</td>
<td>$1284.00$</td>
<td>$1284.03$</td>
</tr>
</tbody>
</table>

Use the compound interest formula with the following problems:

Example 1. Suppose that $1000$ is invested at $6\%$ interest compounded continually. How much would be in the bank after 5 years? How would it compare with $1000$ compounded annually?  
continually $1349.86$ and annually $1276.28$

Example 2. An investor has $5000$ to invest. With which plan would he earn more: Plan A or Plan B?

Plan A: $7.5\%$ compounded annually over a 10 year period?
Plan B: $7\%$ compounded quarterly over a 10 year period?

Solution: Plan A. It would earn $10,305.16$. Plan B would earn $10,007.99$ during the same period.

Hand out the Continuous Growth and the Number $e$ BLMs. Let the students work in groups.

**Activity 3: Applications of Exponential Functions: (GLEs: 10, 24)**

Materials List: Saving for Retirement BLM, pencil, paper, calculator

Saving for retirement is an excellent application of the use of compound interest. Use SQPL- student questions for purposeful learning (view literacy strategy descriptions) to
set the stage for a discussion on the best methods of putting away money for retirement. In general, an _SQPL_ lesson is one that is based on a statement that would cause the students to wonder, challenge, and question. The statement does not have to be factually true as long as it provokes interest. Instead of a statement, use the following scenario to provoke student questions. The activity is designed to teach students the importance of starting at an early age to save for retirement. Instead of a statement, use the following scenario to provoke student questions. It is a superb use of the exponential function in a problem that will teach students more than just the mathematics involved. Hand out the Saving for Retirement BLMs to each of the students in the class and read with them the following:

Two friends, Jack and Bill, both begin their careers at 21. By age 23 Jack begins saving for retirement. He is able to put $6,000 away each year in a fund that earns on the average 7% per year. He does this for 10 years, then at age 33 he stops putting money in the retirement account. The amount he has at that point continues to grow for the next 32 years, still at the average of 7% per year. Bill on the other hand doesn’t start saving for retirement until he is 33. For the next 32 years he puts $6000 per year into his retirement account that also earns on the average 7% per year. At age 65 Jack will have the greatest amount of money in his retirement account.

Have the students work with a partner. Each set of partners should generate one good question about this statement. There will be questions of disbelief, since Jack only puts $60,000 of his own money in his account while Bill puts $192,000 in his account. Some other questions might be

- How is it possible that Jack will have the most money in his retirement fund?
- How much money will Jack have in his account at age 33?
- Does the exponential growth formula work in this case?
- Is there a formula for this type of saving?
- What if the interest rate changes?

Write all the questions on the board so everyone can see them. If a question is asked more than once put a check by it. Add your own questions if students leave out important ones. Once the questions have been asked, encourage the students to figure out how much each person has saved. Students usually have little trouble figuring the amounts for Jack and Bill using their calculator. They also find out that the exponential growth formula cannot be used if money is being added to the account on a regular basis. Once the students realize that the statement is true and they have found the amount of money each will have at age 65, give them the future value formula $F = P \left[ \left( 1 + \frac{i}{i} \right)^n - 1 \right]$ and show them how it is used. Try some different scenarios with the class such as:

(1) What if the interest rate averages 10% instead of 7%, how much Jack and Bill will each have.
(2) Suppose Jack retires at 62. How does his retirement fund compare to Bill’s who will retire at 65?

Students find it hard to believe that by retiring at 62, Jack’s retirement will be so much less. Graph the function $82898(1.07)^x$ and run the table with TblStart at 1 and $\Delta$Tbl=1. This will give the students a chance to see how money grows. The work for the statement is found on the Saving for Retirement with Answers BLM.

**Activity 4: A Look at $\ln x$ Its Local and Global Behavior and Translations in the Coordinate System (GLE 4, 6, 7, 8, 16, 25)**

Materials List: The Local and Global Behavior of $\ln x$ BLM, Translations, Dilations, and Reflections of $\ln x$ BLM, graphing calculator, graph paper, pencil

Vocabulary to be covered for this activity: natural log function.

Look again at the graphs of $y_1 = 2^x$, $y_2 = e^x$, $y_3 = 3^x$. Notice that each of the graphs is strictly increasing and therefore passes the horizontal line test. Each must have an inverse. In this activity the natural log function is introduced. Students learn that it is the inverse function of $f(x) = e^x$. Students should see this algebraically, numerically, and graphically. Remind them that the domain of the function is the range of its inverse, and the range of the function is the domain of the inverse. Demonstrate this using the graphing calculator:

$$e^{(2)} \approx 7.389056099$$
$$\ln(Ans) \approx 2$$

$\ln(\text{Ans})$ must be used since $e^2$ is an irrational number. The point $(0, 1)$ is the only rational ordered pair for $f(x) = e^x$, and $(1, 0)$ the only one for $f(x) = \ln x$. If the domain of $y = e^x$ is the set of reals, then the range of $y = \ln x$ is also the set of reals. The graph of $y = e^x$ uses the negative $x$-axis as a horizontal asymptote, so the graph $y = \ln x$ will use the negative portion of the $y$-axis as a vertical asymptote.

The first blackline master, The Local and Global Behavior of $\ln x$, is a graphing utility activity designed to help students understand why the range of the natural logarithmic function is the set of reals. Most students do not realize the limitations of the calculator graph; therefore, they assume that because the graph stops, that is the minimum value of the range. Hand out The Local and Global Behavior of $\ln x$ BLM. This is best done
individually. Check this blackline master with the class. Make sure that each student understands the domain and range of \( f(x) = \ln x \).

The second blackline master, Translations, Dilations, and Reflections of \( \ln x \) BLM, begins with a modified opinionnaire (view literacy strategy descriptions) that will enable the student to predict how changes in the constants \( a, b, \) and \( c \) will cause changes in the graph of the function \( f(x) = a [\ln(x + b)] + c \). Hand out the Translations, Dilations, and Reflections of \( \ln x \) BLM. Ask students to read each statement. In the column marked My Opinion, students should place a \( \sqrt{\ } \) if they agree with the statement and an X if they disagree. If they disagree, they should explain why in the next column. This portion of the activity should be done independently. Once the students have completed the opinionnaires have them take out their calculators and graph each function to check their work. Go over each of the problems answering any questions that might arise. Finally, give them time to work Part II of the Translations, Dilations, and Reflections of \( \ln x \) BLM. Students can work in groups with Part II.

**Activity 5: Working with the Laws of Logarithms (GLEs: 2, 3)**

Materials List: Working with the Laws of Logarithms BLM, pencil

Vocabulary to be covered for this activity: logarithm to base \( a \), common logarithms, laws of logarithms

Students were introduced to the laws of logarithms in Unit 6 of Algebra II. How much of a review they will need depends on how well they did on questions 6 and 7 of the pretest for this unit. The change of base formula for logarithms: \( \log_a c = \frac{\log_b c}{\log_b a} \) should be covered now. The calculator will give values for common logarithms and logarithms to the base \( e \) only. If they are to evaluate the logarithms they must change to one of the two bases.

Example problems to use for this activity:

1. \( \log_3 4 = \frac{\log 4}{\log 3} = \frac{\ln 4}{\ln 3} \approx 1.261859507 \)

2. Solve: \( 5^{-x} = 20 \) Write the answer to the nearest thousandth.
   \[
   \log(5^{-x}) = \log 20 \\
   -x \log 5 = \log 20 \\
   -x = \frac{\log 20}{\log 5} \\
   -x = 1.457 \\
   x = -1.457
   \]
Hand out Working with the Laws of Logarithms BLM. Students should work individually on this worksheet during class.

Activity 6: Working with Exponential and Logarithmic Functions (GLEs: 4, 6, 7, 25, 27, 28)

Materials List: Working with Exponential and Logarithmic Functions BLM, graph paper, graphing calculator, pencil

This activity will revisit some of the concepts in Unit 1, this time using exponential and logarithmic functions. Begin by having students work the following problems as a review:

1. Begin with the graph of \( y = 2^x \). Give the students the equation \( y = 12 - x \). Ask the students how the graph would differ?
   Below is the graph of \( y = 2^x \) with \( y = 2^{x-1} \).

   The graph of \( y = 2^{x-1} \) shifts one unit left.

   ![Graph](image)

   Ask the students to give the equation needed to reflect \( y = 2^{x-1} \) through the \( x \)-axis.
   **Answer:** \( y = -2^{x-1} \)

2. Given the function \( h(x) = e^{x^2} \) and \( h(x) = f(g(x)) \):
   a) Identify \( f(x) \) and \( g(x) \). Give the domains and ranges of each of the three functions.
   b) Classify the composite function \( h(x) = e^{x^2} \) as even, odd, or neither.

   **Solutions:**
   a) \( f(x) = e^x \) and \( g(x) = x^2 \). The domain of each of the functions is the set of reals. The range of \( h(x) \) is \( \{ y: y \geq 1 \} \). The range of \( f(x) \) is \( \{ y: y > 0 \} \) and the range of \( g(x) \) is \( \{ y: y \geq 0 \} \).
   b) \( h(x) \) and \( g(x) \) are both even functions while \( f(x) \) is neither.

Hand out the Working with Exponential and Logarithmic Functions BLMs. Students should first work on it by themselves, then check their answers with their groups. Encourage them to sketch the graph by hand before looking at it on the calculator.
Activity 7: Solving Exponential Equations (GLEs: 6, 7, 10)

Materials List: Solving Exponential Equations BLM, paper, pencil, graphing calculator

This activity is designed to give students practice in solving the equations they will encounter in the exponential growth and decay problems. Part A of this activity is non-calculator based. Students will need to use the laws of exponents and logarithms to write the exact answer. Part B of the activity uses a calculator. The equations are the type students will encounter in solving real-life problems. Part C uses graphs to solve the problem.

Problems to use as examples in class:
1. $2200 = 300(1.05)^t$
2. $3^{-4} = 10$
3. Use a graphing utility to solve the equation: $e^{x-1} = 2 + 4x$

Solutions:
1. $\frac{\ln 22/7}{\ln 1.05} = 23.471$ to the nearest thousandth
2. $6.096$ to the nearest thousandth
3. 

Activity 8: Problems Involving Exponential Growth and Decay (GLE 2, 3, 10, 24)

Materials List: Applications Involving Exponential Growth and Decay BLM, paper, pencil, graphing calculator

Vocabulary to add to the list: half-life.

Students will be working again with variations of the exponential growth/decay equation $P = P_o a^t$ where $t$ is used to represent time. In the applications shown here, the formula is written in another form $P = P_o (a)^{kt}$ where $k =$ time needed to multiply $P_o$ by $a$. For instance:
• A financial planner tells you that you can double your money in 12 years. This would be written as $P = P_0(2)^{t/12}$. Note that $P(12) = 2P_0$.

• The half-life of a radioactive isotope is given as 4 days. This would be written as $P = P_0\left(\frac{1}{2}\right)^{t/4}$. Note that $P(4) = \frac{1}{2}P_0$.

Hand out Applications Involving Exponential Growth and Decay BLMs. Students should work in their groups on this activity. See Specific Assessments: Activity 8 in the Sample Assessments section for scoring suggestions.

Activity 9: Adding to the Function Portfolio (GLE 3, 4, 6, 7, 10, 19, 27)

Materials List: Library of Functions – The Exponential Function and Logarithmic Function BLM, pencil, paper, graph paper

Exponential and logarithmic functions should be added to the Library of Functions at this time. Hand out the Library of Functions – The Exponential Function and Logarithmic Function BLM to each student. For each function, students should present the function in each of the 4 representations. They should consider

- domain and range
- local and global characteristics such as symmetry, continuity, whether the function has local maxima and minima with increasing/decreasing intervals or is a strictly increasing or strictly decreasing function with existence of an inverse, asymptotes, end behavior, concavity
- the common characteristics of exponential functions such as the constant rate of change, the existence of a y-intercept, and a zero in all functions where the slope $\neq 0$.
- examples of translation, reflection, and dilation in the coordinate plane
- a real-life example of how the function family can be used showing the 4 representations of a function along with a table of select values

The work they have done in this unit should provide them with the necessary information.

Sample Assessments

General Assessments

• The students will perform a writing assessment. They have added to their glossary notebook throughout this unit. They have also had to explain answers with many of their activities. Therefore, one of the assessments should cover this material. Look for understanding of how the term or concept is used. Use verbs such as
explain, show, describe, justify, or compare and contrast. Some possible topics are given below:

- Explain to a friend how to rewrite the algebraic form of an exponential function \( y = a^x \) as a logarithmic function with base \( a \). What is the relationship of these two functions?
- How are the domains and ranges of the functions defined by \( y = e^x \) and \( y = \ln x \) related?
- Compare an exponential growth model to an exponential decay model. Give examples of each.

- Students should complete four “spirals” during this unit. A weekly review of previously learned concepts should be ongoing. One of the favorite methods is a weekly “spiral”, a handout of 10 or so problems covering work previously taught in the course. They can be tied to the study guide for a unit test or as part of a review for the midterm or final exam. The first spiral for this unit should cover the material covered in the Algebra II course. An example of such a spiral is the Spiral BLM. Design other spirals that review what students missed on the pretest as well as using problems that (1) reinforce the concepts learned in earlier activities, (2) review material taught in earlier courses, or (3) review for the ACT or SAT.
- The student will turn in this entry for the Library of Functions for an assessment using the Library of Functions – The Exponential Function and Logarithmic Function BLM.

**Activity-Specific Assessments**

- **Activity 1**: Students should demonstrate proficiency in working with data that is exponential. Give students a set of data, ask them to graph it then find the exponential functions that model the data.

- **Activity 7**: Students should demonstrate proficiency in solving exponential and logarithmic equations. Have the students explain how to solve at least one.

- **Activity 8**: The students should work with a group on problems based on the real-life problems using exponential or logarithmic functions. The scoring rubric should be based on
  1. teacher observation of group interaction and work
  2. explanation of each group’s problem to class
  3. work handed in by each member of the group
Advanced Math – Pre-Calculus
Unit 4: Trigonometry of Triangles

Time Frame: 3.5 weeks

Unit Description

This unit concentrates on Triangle Trigonometry. There is a review of the right triangle ratios with an emphasis on learning how to solve real-life problems using those ratios. The Laws of Sine and Cosine are presented so that problems involving oblique triangles can be solved. Since these laws are used extensively in aviation and physics in conjunction with vectors, operations with vectors are also included in the unit.

Student Understandings

Students will learn to solve triangles using various combinations of sides and angles. They will use Triangle Trigonometry to model and solve real-life problems. Vectors play a part in many of the real-life models involving Triangle Trigonometry. Students will be able to perform basic vector operations, to represent vectors graphically, and to apply their knowledge to problems involving navigation.

Guiding Questions

1. Can students solve real-life problems involving right triangles?
2. Can students use the fundamental trigonometric identities?
3. Can students use the Law of Cosines and Law of Sines to model and solve real-life problems?
4. Can students find the areas of oblique triangles?
5. Can students represent vectors as directed line segments?
6. Can students write the component form of vectors?
7. Can students add, subtract, multiply and find the magnitude of the vector algebraically?
8. Can students find the direction angles of vectors?
9. Can students use vectors to model and solve real-life problems involving quantities that have both size and direction?
Unit 4 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number and Number Relations</strong></td>
<td></td>
</tr>
<tr>
<td>Grade 10</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>Define sine, cosine, and tangent in ratio form and calculate them using technology (N-6-H)</td>
</tr>
<tr>
<td><strong>Measurement</strong></td>
<td></td>
</tr>
<tr>
<td>Grade 10</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td>Model and use trigonometric ratios to solve problems involving right triangles (M-4-H) (N-6-H)</td>
</tr>
<tr>
<td>Grade 11-12</td>
<td></td>
</tr>
<tr>
<td>11.</td>
<td>Calculate angle measures in degrees, minutes, and seconds (M-1-H)</td>
</tr>
<tr>
<td>14.</td>
<td>Use the Law of Sines and the Law of Cosines to solve problems involving triangle measurements (M-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>Grade 10</td>
<td></td>
</tr>
<tr>
<td>12.</td>
<td>Apply the Pythagorean theorem in both abstract and real life settings (G-2-H)</td>
</tr>
<tr>
<td>Grade 11-12</td>
<td></td>
</tr>
<tr>
<td>16.</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
</tbody>
</table>

The first two activities in this unit will review material covered in geometry. Students have covered the sine, cosine, and tangent ratios, their uses and applications in Unit 5 in the geometry course. Special right triangles are also covered in that course. The Pretest Triangle Trigonometry BLM for this unit is designed to discover how much students remember.

Sample Activities

Ongoing: Glossary

Materials List: index cards, What do You Know About Triangle Trig and Vectors? BLM, pencil

Continue the use of the glossary activity in this unit. Students will repeat the two methods used in Units 1, 2, and 3 to help them understand the vocabulary for Unit 4. Begin by having each student complete a self-assessment of his/her knowledge of the terms using a modified vocabulary self awareness chart (view literacy strategy descriptions), What Do You Know About Triangle Trig and Vectors? BLM. Do not give the students definitions or examples at this stage. Ask the students to rate their understanding of each term with a “+” (understand well), a “?” (limited understanding or unsure) or a “–” (don’t know). Over the course of the unit students are to return to the chart, add information or examples, and re-evaluate their understandings of the terms or concepts.
Students should continue to add to their vocabulary cards (view literacy strategy descriptions) introduced in Unit 1. Make sure that the students are staying current with the vocabulary cards. Time should be given at the beginning of each activity for students to bring them up to date.

Students should add to their glossary the following terms as they are encountered in the unit: sine, cosine, tangent, secant, cosecant, cotangent, exact value, angles of elevation and depression, line of sight, degree, minute, second used as angle measurement, oblique triangles, Law of Sines, Law of Cosines, vector, initial point, terminal point, vector in standard position, unit vectors, zero vector, equal vectors, magnitude of a vector, scalar, horizontal, and vertical components of a vector, bearing, heading, air speed, ground speed, true course

Activity 1: Solving Right Triangles (GLEs: Grade 10: 3, 8, 12, Grade 11/12: 11)

Materials List: Solving Right Triangles BLM, pencil, paper, calculator

Students need to be familiar with the following vocabulary for this activity: sine, cosine, tangent, secant, cosecant, cotangent, exact value

This activity is a review of Right Triangle Trigonometry studied in Geometry as well as an introduction to the other trigonometric ratios and the fundamental identities. Students should be able to
- identify the side opposite and the side adjacent to an angle in a right triangle
- identify an included angle
- identify the side opposite an angle
- identify and use sine, cosine, tangent, secant, cosecant, and cotangent
- work with the Pythagorean Theorem
- work with the fundamental identities

This is also an ideal time to review with the students some of the properties of geometric figures that might be used as part of a Triangle Trigonometry problem. Students should review the properties of the 30-60-90 degree and 45-45-90 degree triangles and be able to give the exact values for each of the trig ratios for those angles. They should know how to find the exact values of the sine, cosine, and tangent ratios for each of those angles.

They should also have practice in finding the angle when the ratio is given. This is also the time to introduce them to the fundamental identities. Use the Pythagorean Theorem to develop $\sin^2 \theta + \cos^2 \theta = 1$, then the reciprocal and quotient identities to develop $1 + \tan^2 \theta = \sec^2 \theta$ and $1 + \cot^2 \theta = \csc^2 \theta$. 
The Fundamental Trigonometric Identities

The Reciprocal Identities
\[
\sin \theta = \frac{1}{\csc \theta} \quad \cos \theta = \frac{1}{\sec \theta} \quad \tan \theta = \frac{1}{\cot \theta} \quad \csc \theta = \frac{1}{\sin \theta}
\]

The Quotient Identities
\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \text{and} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}
\]

The Pythagorean Identities
\[
\sin^2 \theta + \cos^2 \theta = 1 \quad 1 + \tan^2 \theta = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta
\]

Give the students the Solving Right Triangles BLMs as a classroom exercise. The activity asks that students give the answer as an exact value which means to write the answer in simplest radical form. At this writing there are universities that give the placement/credit tests in Trigonometry without calculator. Therefore it is important that students review simplification of radicals. Standardized tests will have the multiple choice answers in simplest radical form.

Activity 2: Solving Right Triangles Using Real-life situations (GLEs: Grade 10: 3, 8, 12, Grade 11/12: 11)

Materials List: Right Triangles in the Real World BLM, pencil, paper, calculator

Students need to be familiar with the following vocabulary for this activity: degrees, minutes, and seconds as used in angle measurement, angle of elevation, angle of depression, line of sight

This activity gives students a chance to see how the process of solving a right triangle can be used in real-life. Angles of depression usually give students some trouble. The writing exercise is designed to help students clarify in their minds as to how an angle of depression is constructed.

This would also be a good time to review significant digits with the students and to incorporate angle measures.

A length measured to 1 significant digit corresponds to 10°
A length measured to 2 significant digits corresponds to 1°
A length measured to 3 significant digits corresponds to 0.1° or 10’
A length measured to 4 significant digits corresponds to 0.01° or 1’

When you multiply or divide measurements, round the answer to the least number of significant digits in any of the numbers.
Example 1:

When the angle of elevation of the sun is 53°, a tree casts a shadow 6.5 meters long. How tall is the tree?

Solution:

Let $x$ be the height of the tree. Then $\tan 53^\circ = \frac{x}{6.5}$

$x = 6.5(\tan 53^\circ)$

$= 8.625...$

$= 8.6$ meters tall

*In view of the accuracy of the given data, the answer should be 2 significant digits.*

Example 2:

A ladder 4.20 meters long is leaning against a building. The foot of the ladder is 1.75 meters from the building. Find the angle the ladder makes with the ground and the distance it reaches up the building.

Solution:

Let $x =$ the distance the ladder reaches up the building

$x^2 + 1.75^2 = 4.20^2$

$x = 3.82$ meters

Let $a =$ the angle the ladder makes with the ground

$\cos a = \frac{1.75}{4.20}$

$a = \cos^{-1}(0.416)$

$a = 65.4^\circ$ or $65^\circ 20'$

Hand out Right Triangles in the Real World BLMs. Students should

- Draw a picture and identify the known quantities
- Set up the problem using the desired trigonometric ratio
- Give their answers in significant digits for the angles and sides

This is an excellent activity for group work.

**Activity 3: Solving Oblique Triangles (GLEs: 11, 14)**

Materials List: Solving Oblique Triangles BLM, pencil, paper, calculator

Students need to be familiar with the following vocabulary for this activity: oblique triangles, Law of Sines, Law of Cosines
Mathematicians have proven that a unique triangle can be constructed if given
- two sides and the included angle (SAS)
- three sides (SSS)
- two angles and the included side (ASA)
- two angles and a third side (AAS)

Having the measures of two sides and one angle (SSA) will not necessarily determine a triangle. Given two sides and the included angle or three sides, students will use the Law of Cosines. Given two angles and the included side or two angles and the side opposite one of the angles, students will use the Law of Sines. The Law of Sines is also used with what is called the ambiguous case. This applies to triangles for which two sides and the angle opposite one of them are known. It is called the ambiguous case because the given information can result in one triangle, two triangles, or no triangle. This exercise gives a student practice in (1) determining whether or not the information will produce a unique triangle and (2) selecting which formula to use. Students should understand that they should not use the Law of Sines to find angle measures unless they know in advance whether the angle is obtuse or acute. This activity will utilize a modified word grid (view literacy strategy descriptions). Word grids are utilized to elicit student participation in learning important terms and concepts. This is a modification of the standard word grid in that students fill in cells with more than checks, pluses, or minuses. This word grid will give students practice in
1) applying the geometric postulates of SAS, SSS, ASA, AAS and SSA to the given information
2) choosing the correct formula
3) applying that formula to find the missing sides and/or angles and
4) writing the angles in degrees, minutes, and seconds.

Distribute the Solving Oblique Triangles BLMs. This is an excellent activity for group work. Group members will work together to complete each cell of the BLM. After each group is finished allow the groups to compare answers. Finish this activity by quizzing the students on their application of the postulates and their choices of the formula.

**Activity 4: Real-life Problems Involving Oblique Triangles (GLEs: 11, 14)**

Materials List: Real-Life Problems Involving Oblique Triangles BLM, pencil, paper, calculator

This activity gives students practice in solving real-life problems using the Laws of Sines and Cosines. Areas of triangles are also included since the area formula $A = \frac{1}{2}ab\sin C$ uses the same information needed in the Law of Cosines (SAS). When all three sides are given, the student can use Heros Formula $Area = \sqrt{s(s-a)(s-b)(s-c)}$ where $s$ is the semi perimeter (one half the perimeter).
An excellent source of problems is TRIG-STAR. This is an annual contest conducted by the National Society of Professional Surveyors. Go to their website [http://www.nspsmo.org/trig_star/index.shtml](http://www.nspsmo.org/trig_star/index.shtml) for information. Click on NSPS Trig-Star located in the bar on the left hand side of their home page and choose sample tests. They have 5 sample tests with answer keys. One of the problems from the 2002-03 test is on the Real-Life Problems Involving Oblique Triangles BLM. It is #7.

Students will need to know how to interpret a compass reading. In surveying, a compass reading is usually given as an acute angle from the north-south line towards the east or west as shown below.

\[ \text{N40E} \quad \text{S50W} \]

The Real-Life Problems Involving Oblique Triangles BLM contains several different types of problems that require the use of either the Law of Sines or the Law of Cosines to solve. Below is an example of a surveying problem.

A surveyor needs to find the area of the plot of land described below:
From the edge of Bayou Blue proceed 195 feet due east along Basile Road, then along a bearing of S36°E for 260 feet, then along a bearing of S68°W for 385 feet, and finally along a line back to the starting point.

\[ \text{AB} = \frac{3}{4} \text{"} \]
\[ \text{BC} = 1\text{"} \]
\[ \text{CD} = 1\frac{7}{16}\text{"} \]
From the bearings given, $\angle ABC = 90^\circ + 36^\circ$ or $126^\circ$.

$\angle BCD = 180^\circ - (36^\circ + 68^\circ)$ or $76^\circ$

To find the area of ABCD, divide the quadrilateral into two triangles ABC and ADC.

Area of triangle $ABC$ is $\frac{1}{2} \cdot AB \cdot BC \cdot \sin B$

\[
\frac{1}{2} (195)(260) \sin 126^\circ
\]

$= 20,509 \text{ ft}^2$

To find the area of triangle $ADC$ find (a) AC and (b) $\angle ACD$.

(a) Use the Law of Cosines to find AC.

\[
AC^2 = 195^2 + 260^2 - 2(195)(260) \cos 126^\circ
\]

$= 165,226$

$AC \approx 406 \text{ feet}$

(b) To find $\angle ACD$ use the Law of Sines to find $\angle ACB$.

\[
\frac{\sin ACB}{195} = \frac{\sin 126}{399}
\]

$\angle ACB = 23.3^\circ$

Therefore, $\angle ACD = \angle BCD - \angle ACB$.

$\angle ACD = 76^\circ - 23.3^\circ$

$= 52.7^\circ$

To find the area of triangle ACD:

\[
\frac{1}{2} (AC)(CD) \sin ACD
\]

$= \frac{1}{2} (406)(385) \sin 52.7$

$= 62,179 \text{ ft}^2$

The area of quadrilateral ABCD is 82,688 ft$^2$.

Distribute the Real-Life Problems Involving Oblique Triangles BLMs for the students to work on in their groups.
Activity 5: Practice with Vectors (GLEs: 11, 14, 16)

Materials List: Practice with Vectors BLM, graph paper, ruler, protractor, pencil, calculator

Students need to be familiar with the following vocabulary for this activity: vector, initial point, terminal point, zero vector, equal vectors, magnitude of a vector, scalar, components of a vector, resultant, bearing, heading, true course, ground speed, air speed

Vectors are important tools in the study of problems involving speed and direction, force, work and energy. Vectors have been added to triangle trigonometry in this curriculum guide because so many of the real-life problems in trigonometry use vectors. This activity will cover geometric vectors.

A vector is a quantity that has both magnitude and direction. Force and velocity are examples of vector quantities. The physical quantities of speed, mass, or length are described by magnitude alone. There is no direction involved.

- Geometric Vectors
  To represent a vector use a directed line segment or arrow with a bold faced letter to represent it. A bold faced letter in italics is also used in some text books.

\[ \vec{v} \]

A vector has an initial point and a terminal point.

initial point \hspace{2cm} terminal point

Vectors are equal if and only if they have the same magnitude and the same direction.

- Definition of vector addition
  The sum of two vectors goes from the beginning of the first vector to the end of the second vector representing the ultimate displacement. Their sum is called the resultant vector. In the diagram below the first vector is \( \vec{u} \) and the second vector is \( \vec{v} \). The resultant vector is drawn from the initial point of \( \vec{u} \) to the terminal point of \( \vec{v} \).

\[ \vec{u} + \vec{v} \]

If \( \vec{w} = \vec{v} + \vec{u} \) then \( \vec{u} \) and \( \vec{v} \) are called components of \( \vec{w} \) and \( \vec{w} \) is referred to as the resultant of \( \vec{u} \) and \( \vec{v} \).
Scalar multiplication
To find the product of $rv$ of a scalar $r$ and a vector $v$, multiply the length of $v$ by $|r|$. If $r < 0$, then reverse the direction.

Given the vectors $v$ and $u$ shown below:

Sketch the linear combination of (a) $-u + 3v$ and (b) $u - 2v$. Be sure that the magnitude and direction are carefully measured.

(a) $-u + 3v$

(b) $u - 2v$
The magnitude of a vector is also called the norm of a vector or the absolute value of a vector and written $|v|$ or $||v||$ depending on the textbook used.

- **Properties of vectors:**
  1. $||v|| = 0$ if and only if $v = 0$
  2. $||rv|| = |r| ||v||$
  3. $||u + v|| = ||u|| + ||v||$

### Vectors and Navigation

Be sure that students understand and can use the terms:

- **bearing** - the angle $\theta$, $0^\circ \leq \theta < 360^\circ$, measured clockwise, that vector makes with the north
- **heading** – the bearing of the vector that points in the direction in which a craft, such as a ship or a plane, is aimed
- **true course** – the bearing of the vector that points in the direction in which a craft is actually traveling
- **ground speed** - the speed of a plane relative to the ground
- **air speed** – the speed of a plane relative to the surrounding air, that is, the speed the plane would actually have if there were no wind

The diagram above illustrates a plane with an air speed of 300 km/hr and a heading of 40°. This velocity is represented by $\vec{AB}$. Note that $||\vec{AB}||$ is the air speed. The figure also illustrates a wind with a bearing of 90° shown by the vector $\vec{BC}$. The plane has a true course of 50° shown by $\vec{AC}$. 
Example 1: A heading of 45 degrees

Example 2: A heading of 230 degrees

Finding the resultant of two given displacements
Let \( \mathbf{v} \) represent the vector 7.0 units due west and \( \mathbf{u} \) represent the vector 8.0 units along a bearing of 60 degrees as shown below.

Draw a vector from the initial point of \( \mathbf{v} \) to the terminal point of \( \mathbf{u} \). That vector \( \mathbf{w} \) is the resultant of \( \mathbf{u} \) and \( \mathbf{v} \). 
The length of \( \|w\| \), can be found using the Law of Cosines. The angle between sides with length 7 and 8 is 30°.

\[
\|w\|^2 = 7^2 + 8^2 - 2(7)(8)\cos 30°
\]
\[
= \sqrt{23}
\]
\[
= 4.8
\]

To find the bearing use the Law of Sines to find the angle made by the vectors \( v \) and \( w \)

\[
\frac{\sin 30°}{4.8} = \frac{\sin \theta}{8}
\]
\[
\theta = \sin^{-1}(0.83)
\]
\[
= 123°33'26"
\]

Applying significant digits, we obtain 124°. Note: this is an obtuse angle. The sine value of any angle and its supplement is the same. Remind students that they should determine which angle is the correct one. Subtract 90° from that angle to obtain the bearing.

Subtracting 90° gives 34° as the bearing.

Distribute the Practice with Vectors BLMs as a classroom exercise. Let students work together to complete the activity.

**Activity 6: Vectors and Navigation (GLEs: 11, 14, 16)**

Materials List: Vectors and Navigation BLM, calculator, graph paper, pencil

The purpose of this activity is to give students practice using vectors to solve navigation problems and to work with the vocabulary: bearing, heading, true course, ground speed, and air speed. To do this, students will be asked to create a *story chain* (view literacy strategy descriptions) modeled after the example shown below. A *story chain* is very useful in teaching math concepts, while at the same time giving practice in writing and solving problems. It involves a small group of students writing a story problem. The first
student starts the story. The next student adds a sentence then passes it to the third student to do the same. If a group member disagrees with any of the work or information that has been given, the group discusses the work and either agrees to revise the problem or move on as it is written. Once the story is finished and the solution found, each group shares its story with other groups in the class. A new group will work the problem and then check with the group who wrote the problem to determine if its solution is correct. In this particular activity students will write a navigation problem. Each student will take a part and provide the solution for the part he or she writes. Once the problem is completed and the solution agreed upon the group will challenge other groups to solve their problem.

The model problem
Part One: The airspeed of a light plane is 200 mph and is flying on a heading of 90° (due east) from Baton Rouge, LA, to Tallahassee, FL. A 40 mph wind is blowing with a bearing of 160°. The vector representation is shown below:

Part Two: Find the ground speed of the airplane. In the diagram below
- \( u \) represents the velocity of the plane relative to the air
- \( v \) the velocity of the plane relative to the ground
- \( w \) the wind velocity
- \( \theta \) the true course

To find the ground speed use the Law of Cosines: \( ||v||^2 = u^2 + w^2 - 2uw \cos \alpha \). The angle \( \alpha \) is 110° (90° + 20°)

Solution:
\[
||v||^2 = 200^2 + 40^2 - 2(200)(40)\cos110^\circ
\]
\[
v = 217 \text{ mph}
\]
Part Three: Because of the wind the true course of the airplane will not be the heading. Find the true course of the airplane. To do this find $\beta$, the angle made by vectors $\mathbf{u}(\text{theairspeed})$ and $\mathbf{v}(\text{the ground speed})$, using the Law of Sines.

Solution:

\[
\sin \beta = \frac{\sin 110^\circ}{217} \\
\sin^{-1} \left( \frac{40 \sin 110^\circ}{217} \right) = 10.8^\circ \text{ or } 10^\circ 48'
\]

The true course is $90^\circ + 10^\circ 48'$

$100^\circ 48'$

Part Four: What heading should the pilot set so that the true course is $90^\circ$? Draw a new diagram.

\[
\mathbf{u}' = \text{airspeed } (200 \text{ mph}) \\
\mathbf{v}' = \text{groundspeed} \\
\beta = \text{angle made by vectors } \mathbf{u}' \text{ and } \mathbf{v}' \\
\alpha = \text{angle made by vector } \mathbf{u}' \text{ and } \mathbf{w}, \text{the wind vector}
\]

Solution:

To find $\theta$ use the Law of Sines

\[
\sin \beta = \frac{\sin 70}{200} \\
\therefore \beta = \sin^{-1} \left( \frac{40 \sin 79}{200} \right) = 10.8^\circ
\]

Then $\theta = 90^\circ - 10.8^\circ = 79.2^\circ$

The pilot should change course to $79.2^\circ$ or $79^\circ 10'$ in order to reach Tallahassee, FL.

The story chain will work as follows:
Divide the students into groups of four. Each group is to write a problem, find its solution, and then challenge the other groups in the class to solve the problem. There is a set of directions and a page to record the problem in the blackline master section. Give
each student in the group both of these sheets. Each person in the group is to record the problem. Use one of them for the group solution and use the other three to pass out for the challenge.

Student #1 will begin the story picking the starting city and ending city, finding the latitude and longitude for each, and determining the direction in which the plane will fly. He or she will then choose an airplane from the list found on the Vectors and Navigation BLM to determine the air speed of the plane and fill in the information on the story page. The student should then begin the solutions page drawing and labeling the vectors for airspeed and wind speed.

Student #2 will draw in the vector representing the ground speed of the airplane, find the angle formed by \( \mathbf{u} \) (the velocity of the plane) and \( \mathbf{w} \) (the wind velocity), then find the ground speed. All work should be shown on the solutions page.

Student #3 will find the true course of the airplane and put his or her work on the solutions page.

Student #4 will determine the heading needed so that the plane will fly on the needed course to reach the correct destination. A vector diagram should be drawn showing \( \theta \), the needed course correction.

Hand each student a copy of Vectors and Navigation BLM and go over the directions for creating the problem. Explain what is expected of each person in the group.

**Activity 7: Algebraic Representation of Vectors (GLEs: 11, 14, 16)**

Materials List: Algebraic Representation of Vectors BLM, pencil, paper, graph paper, calculator

Students need to be familiar with the following vocabulary: horizontal and vertical components of a vector, vector in standard position, unit vectors

In the coordinate system below, the vector \( \mathbf{v} \) consists of a 3 unit change in the x-direction and a -2 unit change in the y direction.

![Vector Diagram](image)

The numbers -2 and 3 are called components of \( \mathbf{v} \). We can write \( \mathbf{v} = (3, -2) \). Since vectors with the same magnitude and direction are equal we can draw an infinite number of vectors with the components of \( (3, -2) \).
Given: \(A(x_1, y_1)\) and \(B(x_2, y_2)\). The components of \(\overrightarrow{AB}\) can be found by subtracting the coordinates of \(A\) and \(B\).

\[
\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)
\]
and the components are \(x_2 - x_1\) and \(y_2 - y_1\). The magnitude is found by using the formula for distance between two points:

\[
\| \overrightarrow{AB} \| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

**Example 1:**

Given \(A(-1, 3)\) and \(B(3, -1)\).

a) Find the components and express \(\overrightarrow{AB}\) in component form.
b) Find the magnitude.

a) The components are 4 and -4 and \(\overrightarrow{AB} = (4, -4)\)
b) \(\| \overrightarrow{AB} \| = 4\sqrt{2} \approx 5.7\)

The following operations are defined for 
\(\vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2)\), and any real number \(k\).

- Addition : \(\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2)\)
- Subtraction: \(\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2)\)
- Scalar multiplication: \(k\vec{a} = (ka_1, ka_2)\)

**Example 2:**

Given \(\vec{a} = (5,2)\) and \(\vec{b} = (1,3)\).

Find:

(1) \(\vec{a} + 2\vec{b}\)

Solution

\[
\begin{align*}
(5, 2) + 2(1,3) \\
= (5, 2) + (2, 6) \\
= (7, 8)
\end{align*}
\]
Verify this geometrically using graph paper. Draw $\vec{a}$ and $2\vec{b}$ with their initial points at the origin of a coordinate system. Then slide $2\vec{b}$ so that its initial point is at the terminal point of $\vec{a}$. Draw the resultant $\vec{a} + 2\vec{b}$. Note that its terminal point is at the point (7,8).

**Example 3:** Suppose an airplane is climbing with a horizontal velocity of 325 miles per hour and a vertical velocity of 180 miles per hour. Let $\vec{i}$ and $\vec{j}$ be unit vectors in the horizontal and vertical directions. In this problem each has a velocity of 1 mph. We can then say that the velocity vector $\vec{v}$ is the sum of the horizontal velocity $325\vec{i}$ and the vertical velocity $180\vec{j}$, so $\vec{v} = 325\vec{i} + 180\vec{j}$. The $325\vec{i}$ and $180\vec{j}$ are called the horizontal and vertical components of $\vec{v}$. What is the velocity of the airplane?

$$||\vec{v}|| = \sqrt{(325^2 + 180^2)}$$

$$= 372 \text{ mph}$$

**Example 4:** Vector $\vec{a}$ has a magnitude 4 and a direction $45^\circ$ from the horizontal. Resolve $\vec{a}$ into horizontal and vertical components.

Let $(x, y)$ be the point at the head of $\vec{a}$. By the definitions of sine and cosine,

$$\frac{x}{4} = \cos 45^\circ \quad \text{and} \quad \frac{y}{4} = \sin 45^\circ$$

Therefore, $x = 2.83$ and $y = 2.83$.

So $\vec{a} = 2.83\vec{i} + 2.83\vec{j}$

➢ If $\vec{v}$ is a vector in the direction $\theta$ in standard position, then $\vec{v} = x\vec{i} + y\vec{j}$

where $x = ||\vec{v}||\cos \theta$ and $y = ||\vec{v}||\sin \theta$

**Example 5:** Vector $\vec{a}$ is 6 at $40^\circ$ and vector $\vec{b}$ is 4 at $110^\circ$. Find the resultant $\vec{r}$ as:

a) The sum of two components

b) A magnitude and direction angle

**Solution:**

a) $\vec{r} = \vec{a} + \vec{b}$

$$= (6\cos 40^\circ) \vec{i} + (6\sin 40^\circ) \vec{j} + (4\cos 110^\circ) \vec{i} + (4\sin 110^\circ) \vec{j}$$

$$= (6\cos 40^\circ + 4\cos 110^\circ) \vec{i} + (6\sin 40^\circ + 4\sin 110^\circ) \vec{j}$$

$$= 3.228...\vec{i} + 7.615...\vec{j}$$

$$= 3.23\vec{i} + 7.62\vec{j}$$

b) $||\vec{r}|| = \sqrt{(3.228)^2 + (7.615)^2}$

$$= 8.27$$
\[ \theta = \tan^{-1}\left(\frac{7.62}{3.23}\right) \]

\[ = 67.0^\circ \]

Hand out the Algebraic Representation of Vectors BLMs. Let the students work in their groups.

**Sample Assessments**

**General Assessments**

- A writing assessment should be assigned for the unit. Students have added to their notebook glossary throughout this unit. They have also had a short writing assignment with each of their activities. Therefore, one of the assessments should cover this material. Look for understanding of how the term or concept is used. Use verbs such as show, describe, justify, or compare and contrast.

- Students should complete three “spirals” during this unit. Review of previously learned concepts should be ongoing throughout the unit. One of the favorite methods is a weekly “spiral”, a handout of 10 or so problems covering work previously taught in the course. Tie them to the study guide for a unit test or a midterm exam. The Spiral BLM is an example of one on Right Triangle Trigonometry (Activity 1). Another should give students more practice on vectors.

- The students should be divided into groups to work the problems. Choose a problem or problems that require more work than the textbook problems. They can be problems such as those put out by the National Society of Professional Surveyors in their annual TRIG-STAR contest. This is a contest based on the practical application of Trigonometry. The website is [http://www.nspsmo.org/trig_star/index.shtml](http://www.nspsmo.org/trig_star/index.shtml) Give each group a different problem.

Scoring rubric based on
1. teacher observation of group interaction and work
2. explanation of each group’s problem to class
3. work handed in by each member of the group

**Activity-Specific Assessments**

- **Activity 1:** Students should demonstrate proficiency in their understanding and use of the right triangle ratios. Sample questions to use:

  1. In triangle ABC \[ \angle C = 90^\circ, \quad \sin B = \frac{3}{5}, \] and side \( c = 15 \). Find the length of side \( b \).
2. \( \sec A = \frac{13}{12} \). Find the other five trigonometric ratios.

3. In triangle ABC \( \angle C = 90^\circ \), side \( a = 4 \), and side \( b = 8 \). Find \( \angle A \) and \( \angle B \).

4. Geometric shapes containing right triangles that require students to find the missing sides or angles.

5. Trigonometric ratios of the special right triangles (non-calculator).

6. Simplifying or proving identities.

- **Activity 3**: Students should demonstrate proficiency in their ability to determine (a) whether or not the given information will result in a unique triangle and (b) which formula should be used to solve the triangle.

- **Activity 5 and/or 7**: Students should demonstrate proficiency in using vector notation as well as adding and subtracting vectors both algebraically and geometrically. It should resemble these activities and cover questions 4 – 6 found under Guiding Questions.
Advanced Math - Pre-Calculus
Unit 5: Trigonometric Functions

Time Frame: 3.5 weeks

Unit Description

This unit begins with an introduction of angles as defined in trigonometry. The unit circle is then used to define the sine and cosine functions and their graphs. Trigonometric functions of any angle are then evaluated. Graphing using transformations and solving trigonometric equations is covered before students use the functions to model and solve real-life problems.

Student Understandings

Students are able to move between degree and radian measure. They can use radian measure to model real-life problems. They understand the meaning of periodic function and can use it to help evaluate the trigonometric functions. Students are able to find reference angles for any angle in both degrees and radians. They recognize the graphs of the sine and cosine function and can translate them in the coordinate system.

Guiding Questions

1. Can students describe an angle and convert between degree and radian measure?
2. Can students use angles to model and solve real-life problems involving arc length, areas of sectors, linear speed, and angular speed?
3. Can students identify the unit circle and its relationship to real numbers?
4. Can students evaluate the trigonometric functions sine and cosine using the unit circle?
5. Can students use the unit circle to graph the sine and cosine function identifying their domains and ranges?
6. Can students identify the other four trigonometric functions from the fundamental identities?
7. Can students use reference angles to evaluate trigonometric functions?
8. Can students use amplitude, period, phase shift, and vertical displacement to sketch the graphs of sine and cosine? Can they use the same information to write their equations?
9. Can students use trigonometric functions to model and solve real-life problems?
Unit 5 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>6</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics for the function (A-3-H)</td>
</tr>
<tr>
<td>8</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)</td>
</tr>
<tr>
<td><strong>Measurement</strong></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Calculate angle measures in degrees, minutes, and seconds (M-1-H)</td>
</tr>
<tr>
<td>12</td>
<td>Explain the unit circle basis for radian measure and show its relationship to degree measure of angles (M-1-H)</td>
</tr>
<tr>
<td>13</td>
<td>Identify and apply the unit circle definition to trigonometric functions and use this definition to solve real-life problems</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>Grade 10</td>
<td></td>
</tr>
<tr>
<td>13.</td>
<td>Solve problems and determine measurements involving chords, radii, arcs, angles, secants, and tangents of a circle (G-2-H)</td>
</tr>
<tr>
<td>Grade 11/12</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>25</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>28</td>
<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
</tr>
</tbody>
</table>

Sample Activities

**Ongoing: The Glossary**

Materials List: index cards 3 x 5 or 5 x 7, What Do You Know About Trigonometric Functions? BLM, pencil, pen

Students continue the glossary activity in this unit. They will repeat the two methods used in units 1 through 4 to help them understand the vocabulary for Unit 5. Begin by having each student complete a self-assessment of his/her knowledge of the terms, using a modified **vocabulary self awareness** chart ([view literacy strategy descriptions]), What Do You Know About Trigonometric Functions? BLM. Most or all of the terms will be
unfamiliar to the student at this point. Do not give the students definitions or examples at this stage. Ask the students to rate their understanding of each term with a “+” (understand well), a “?” (limited understanding or unsure) or a “–”(don’t know). Over the course of the unit students are to return to the chart, add information or examples, and re-evaluate their understanding of the terms or concepts.

Students should continue to add to their vocabulary cards (view literacy strategy descriptions) introduced in Unit 1. Make sure that the students are staying current with their cards. Time should be given at the beginning of each activity for students to bring them up to date.

Words to add: radian, initial side of an angle, terminal side of an angle, positive angles, negative angles, coterminal angles, quadrantal angles, reference angles, central angles, arc, linear speed, angular speed, unit circle, periodic functions, fundamental period, amplitude, phase shift, sinusoidal axis

Activity 1: Angles in Trigonometry (GLEs: 11, 12)

Materials List: Angles and Their Measure BLM, pencil, calculator

Vocabulary to cover in this activity: radian, initial side of an angle, terminal side of an angle, positive angles, negative angles, coterminal angles, quadrantal angles

Up to this point students have measured angles in degrees $0 \leq \theta < 180^\circ$. This activity introduces students to angles as they are defined in trigonometry. In preparation for the activity cover

- the initial sides and terminal sides of an angle
- angle in standard position (students were introduced to this in the vector sections in Unit 4)
- angle size
- the definition of positive and negative angles
- definition of radian measure
- conversion from radian to degree measure and degree measure to radian measure
- coterminal angles
- quadrantal angles

Fill out the vocabulary cards and look at examples of each one. A modified word grid (view literacy strategy descriptions) will be utilized to help students understand the concepts and definitions listed above. Hand out the Angles and Their Measure BLMs. Let the class work as partners or in groups.
Activity 2: Arcs and Sectors (GLEs: Grade 10: 13; Grade 11/12: 11, 12)

Materials List: Arcs, Sectors, Linear and Angular Speed BLM, pencil, calculator

Vocabulary to cover in this activity: central angles, arcs, angular speed, linear speed

The problems on the Sectors, Arcs, Linear and Angular Speed BLM review the concept of central angles and the arcs they intercept, while introducing the idea of radian measure of an arc. Students learned how to find the lengths of arcs in Geometry using the formula:

\[
\text{Length of the arc} = \frac{x}{360} \cdot 2\pi
\]

where \(2\pi r\) represents the circumference of the circle and \(x\) represents the length of the arc. Use this to show the students that the length of an arc is just the product of an angle measured in radians and the radius of the circle. Advanced Math textbooks give the formula as

\[
s = r \theta
\]

where \(s\) is the length of the arc and \(\theta\) is the angle measured in radians.

Students also learn that the angular speed of a point on a rotating object is the number of degrees, radians, or revolutions per unit time through which the point turns. The formula for the angular speed of that point is:

\[
\text{Angular speed} = \frac{\text{central angle}}{\text{time}} = \frac{\theta}{t}
\]

Linear speed of a point is the distance per unit time that the point travels on its circular path. The linear speed of that point is:

\[
\text{Linear speed} = \frac{\text{arc length}}{\text{time}} = \frac{s}{t}
\]

A point \(P\) moving with constant speed around a circle of radius \(r\) is called uniform circular motion.

Example 1: A particle moves with angular speed of 8 radians per second around a circle with a radius of 6 cm. Find the linear speed in cm/sec.

*The central angle is 8 radians. The radius of the circle is 6 cm. Therefore, the linear speed is 48 cm/sec.*

Example 2: A pulley of radius 4 inches is turning at 8 revolutions per second. Find the angular speed in radians per second. What is the linear speed of the rim of the pulley?

*One revolution is \(2\pi\) so 8 revolutions is \(16\pi\) radians. The angular speed is 16\(\pi\) radians per second or approximately 50.3 radians/sec. The linear speed of the rim of the pulley is approximately 201 inches/sec.*

Students should complete Sectors, Arcs, Linear and Angular Speed BLM as a class exercise. They should be used as a group exercise.
Activity 3: The Unit Circle (GLEs: 4, 8, 12, 13)

Materials List: a large circle superimposed on a coordinate system, Directions for Constructing the Unit Circle BLM, calculator, graph paper, pencil, A Completed Unit Circle BLM

Vocabulary to be covered in this activity: unit circle, fundamental periods of sine and cosine

Part I
Each student should have an accurate unit circle to keep in his or her notebook. Unit circles can be downloaded from the Internet. There are also several excellent interactive sites should the classroom have the needed equipment. However it is to the students’ advantage to construct their own. There should be one in the classroom so that the students can see how the completed circle should look. See A Completed Unit Circle BLM.

Give each student a large circle superimposed on a coordinate system and Directions for Constructing the Unit Circle BLM. Have each student
1. Label the coordinates (1, 0), (0, 1), (-1, 0) and (0, -1) where the circle intersects the coordinate system.
2. Divide the circle into 8 equal arcs. Mark each with the length of the arc, corresponding to values of \(0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \ldots\) and \(0^\circ, 45^\circ, 90^\circ, 120^\circ\ldots\)
3. Find the coordinates of each of those endpoints using the 45°-45°-90° right triangle. Exact values should be put on the circle.
4. Divide the circle into 12 equal arcs corresponding to values of \(0, \frac{\pi}{6}, \frac{\pi}{3}, \ldots\) and \(0^\circ, 30^\circ, 60^\circ\ldots, 360^\circ\).
5. Find the coordinates of each of those endpoints using the 30°-60°-90° right triangle. Exact values should be put on the circle.

This can also be a group activity with groups finding the multiples of \(\frac{\pi}{4}\), \(\frac{\pi}{6}\), or \(\frac{\pi}{3}\) radians and then sharing their findings with the rest of the class. It is helpful for students to see one already made as they construct theirs.

Part II
Once students have completed the unit circle, define the coordinates \((x, y)\) of each point on the circle in terms of \(\theta\).

Let \(\theta\) be a real number and let \((x, y)\) be a point on the circle corresponding to \(\theta\), then \(\sin \theta = y\) and \(\cos \theta = x\).

The four remaining trigonometric functions can also be defined:
Students should use their unit circles and calculators to find other points on the circle. Ask the following questions:

1. What are the endpoints of an arc on this circle that is 2.5 radians in length?
   \( \cos(2.5), \sin(2.5) \approx (-.80, .60) \)

2. What are the endpoints of an arc on this circle that is 4 radians in length?
   \( \cos(4), \sin(4) \approx (-.65, -.76) \)

3. The coordinates of a point on the unit circle are (.96, .28). Locate this point on your circle and find the length of the arc from (1, 0) to this point, going in the counterclockwise direction.
   \( \approx .28 \) radians

4. The coordinates of a point on the unit circle are (.28, -.96). Locate this point on the unit circle and find the length of the arc moving counterclockwise from (1, 0) to the point.
   \( \approx 5 \) radians

Note: Problem #4 is a good problem to show that \( \sin^{-1}(-.96) \) will give a negative angle. Students must add 2\( \pi \) to obtain the angle moving counterclockwise.

**Activity 4: The Sine and Cosine Functions and Their Graphs (GLEs: 4, 6, 8, 11, 12, 13, 25)**

Materials List: graph paper, pencil, graphing calculator

**Part I**

Graphing the Sine and Cosine Functions

Students begin by setting up a table of values for the sine and cosine functions using the unit circle values. Calculators will help in evaluating the radical expressions. Students should use graph paper to construct graphs of the two functions \( f(\theta) = \cos(\theta) \) and \( f(\theta) = \sin(\theta) \) from the table of values. The horizontal axis is labeled \( \theta \) and the vertical axis is labeled \( \cos(\theta) \) or \( \sin(\theta) \). The 16 points should be located on the graph and a smooth curve drawn through them. Have the students identify the five key points on the graph: the intercepts, the maximum point, and the minimum point. This is a good time to point out that with the graph of the sine function, the intercepts are found at the beginning, middle and end of the period. The maximum point is \( \frac{1}{4} \) of the way through the
period and the minimum point is ¾ of the way through the period. With the cosine function the maximum points are at the beginning and end of the period, the minimum point is halfway through the period, and the intercepts are found one-fourth and three-fourths of the way through the period.

They should also note that the points found with the calculator in Part II of Activity 4 lie on the curves they have drawn. The following site does an excellent job showing students the connection between the unit circle and the sine and cosine functions.

Trigonometry Aplet:  http://www.wou.edu/~burtonl/trig.html

Part II
Using the Graphing Calculator
Once students have completed the unit circle activity it is time to look at the graphs of sine and cosine on a graphing calculator. To illustrate the periodic nature of the sine and cosine functions and to compare their common features, have students set MODE to Radian and graph two fundamental periods using the following window:

```
WINDOW
Xmin=0
Xmax=4π
Xscl=1
Ymin=-2
Ymax=2
Yscl=1
Xres=1
```

In TABLE SETUP have ΔTbl move π/12 units.

```
TABLE SETUP
TblStart=0
ΔTbl=π/12
IndPnt: Auto Ask
Depend: Auto Ask
```

This will reinforce the unit circle values. By comparing the table values taken from the unit circle with the table values given by the calculator, the students should see that π/12 covers each of the arcs placed on their unit circles. Use the table feature of the graphing calculator to reinforce the fundamental periods of sine and cosine. Identify the intercepts, maximum points, and minimum points. Students should see that the complete graph of both functions can be drawn in 2π units. By looking at the graphs of sine and cosine, students can visualize what the term periodic function means.
Activity 5: Computing the Values of Trigonometric Functions of General Angles (GLEs: 11, 12)

Materials List: Computing the Values of Trigonometric Functions of General Angles BLM, pencil

Vocabulary to cover: reference angles

Begin by defining the trigonometric functions of any angle. Let $\theta$ be an angle in standard position with $(x, y)$, a point on the terminal side of $\theta$. The distance from the origin to this point is $r$ and $r = \sqrt{x^2 + y^2}$.

(1) $\sin \theta = \frac{y}{r}$

(2) $\cos \theta = \frac{x}{r}$

(3) $\tan \theta = \frac{y}{x}, x \neq 0$

(4) $\cot \theta = \frac{x}{y}, y \neq 0$

(5) $\csc \theta = \frac{r}{y}, y \neq 0$

(6) $\sec \theta = \frac{r}{x}, x \neq 0$

This activity will utilize a modified word grid (view literacy strategy descriptions) to give students practice in
- finding exact values of the trigonometric functions for general angles
- using coterminal angles to find the exact value of a trigonometric function
- being able to determine the sign of a trigonometric function in a given quadrant
- finding the reference angle of a general angle
- finding the exact value of a trigonometric function of an angle given one of the ratios and the quadrant in which the angle is located

Review the vocabulary introduced in Activity 1 before handing out the Computing the Values of Trigonometric Functions of General Angles BLMs. Use these for individual class work or homework. After the grid is filled in, students may work with partners to quiz each other over the content in preparation for tests and other class activity.

Activity 6: The Family of Functions - Sine and Cosine (GLEs: 4, 6, 8, 12, 16, 24, 25, 28)

Materials List: The Family of Functions - Sine and Cosine Part I BLM, The Family of Functions-Sine and Cosine Part II BLM, pencil, graphing calculator

Vocabulary to cover: amplitude, period, phase shift, sinusoidal axis

In this activity, students will study the families of the sine function, $f(x) = \sin x$, and the cosine function, $f(x) = \cos x$. Students have been exposed to the vertical and horizontal stretches (dilations) as well as the vertical and horizontal translations in the families that
they have studied so far. With periodic functions, they will learn a new vocabulary in describing the dilations and translations of each parent function. Do not offer any explanations of the vocabulary or review of translations and dilations until students have completed The Family of Functions - Sine and Cosine Part I BLMs.

In Part One, the students will begin with a SPAWN (view literacy strategy descriptions) writing prompt. SPAWN is an acronym that stands for five categories of writing prompts: Special Powers, Problem Solving, Alternative Viewpoints, What If, and Next. Students will answer a “next” writing prompt. In looking at the family of sinusoidal functions 

\[ f(x) = A \sin(Bx - C) + D \]

and 

\[ f(x) = A \cos(Bx - C) + D \]

students will be asked to write in anticipation of what each of the values A, B, C, and D will do to change the graphs of the parent function. In their writing, students should explain the logic of what they think will happen when the value is used. Hand out The Family of Functions - Sine and Cosine Part I BLM

Once the students have finished their writings, discuss with them what they anticipate the values of A, B, C, and D will do to the graph. Examples using the graphing calculator can illustrate the changes. A table such as the one below helps students organize the needed information for the sine or cosine functions.

<table>
<thead>
<tr>
<th>Method I</th>
<th>Method II</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ f(x) = A \sin(Bx - C) + D ]</td>
<td>[ f(x) = A \sin(Bx - C) + D ]</td>
</tr>
<tr>
<td>The amplitude is (</td>
<td>A</td>
</tr>
<tr>
<td>(B) causes a horizontal stretch or shrink. The period of the function is (\frac{2\pi}{B}).</td>
<td>(B) causes a horizontal stretch or shrink. The period of the function is (\frac{2\pi}{B}).</td>
</tr>
<tr>
<td>The phase shift is (C). (C) causes a horizontal translation of the graph. The phase shift is to the left if (C &lt; 0); to the right if (C &gt; 0).</td>
<td>The phase shift is (\frac{C}{B}). This causes a horizontal translation of the graph. The phase shift is to the left if (C &lt; 0); to the right if (C &gt; 0).</td>
</tr>
<tr>
<td>The vertical translation is (D). This causes the sinusoidal axis to be shifted (D) units. If (D &lt; 0) the axis shifts down. If (D &gt; 0) the axis shifts up.</td>
<td>The vertical translation is (D). This causes the sinusoidal axis to be shifted (D) units. If (D &lt; 0) the axis shifts down. If (D &gt; 0) the axis shifts up.</td>
</tr>
</tbody>
</table>
Example 1:

Find the amplitude and sinusoidal axis for \( y = -2 \cos(x) + 3 \). What is the range of the function?

*The amplitude is \(|-2| = 2\).* The vertical translation is 3 units so the sinusoidal axis is now \( y = 3 \). The range is \([1, 5]\).

Example 2:

Find the period and phase shift for \( y = \cos\left(\frac{1}{4}x - \pi\right) \).

*Use the formula \( \frac{2\pi}{B} \) to find the period. The period is \( 8\pi \). The phase shift is \( \frac{C}{B} \).*

In this equation the phase shift is \( 4\pi \) units to the right.

Example 3:

Find the amplitude, period, phase shift, and sinusoidal axis given the equation

\[ y = 3 \sin\left(\frac{\pi}{4} x + \frac{3\pi}{4}\right) + 1 \]

What is the range of the function?

*The amplitude is 3.* The period is \( \frac{2\pi}{\frac{\pi}{4}} \) or 8. The phase shift is \( \frac{3\pi}{\frac{\pi}{4}} \) or 3. This gives a horizontal shift of 3 units to the left. The vertical shift is 1, so the sinusoidal axis is \( y = 1 \). The range is \([-2, 4]\).

Hand out The Family of Functions - Sine and Cosine Part II BLMs. Allow students to work in their groups to complete the activities.

**Activity 7: Solving Real-life Problems Using the Trigonometric Functions (GLEs: 4, 6, 8, 13, 24, 25, 28)**

Materials List: Finding Daylight BLM, pencil, graph paper, calculator

There are a number of real-life situations that can be modeled by periodic functions. A partial list is given below.

- tides (Tidal charts can be found at [http://www.saltwatertides.com](http://www.saltwatertides.com))
- satellites orbiting the earth
- amount of daylight throughout the year
- sunspots
- predator prey problems
- Ferris wheels or any object that rotates such as paddle wheels
The Finding Daylight BLM will explore the amount of daylight throughout the year for cities at various latitudes. It is a group activity whereby each student within the group is given a different city. The site used to obtain the data is http://aa.usno.navy.mil/AA/data/docs/RS_OneYear.html#forma.

Additional groups of cities can be put together if there are more than four groups in the class. There are 22,000 locations in the website’s database. Students will use graphing calculators to enter their data and graph their scatterplots. They will also obtain the best fit equation for the data. They are expected to graph scatterplots by hand. Have the students use a common scaling for the graphs so that the cities can be compared visually. Use the data to find the amplitude, period, phase shift, and vertical shift. Finally, they will write sine or cosine equations modeling their data. (While close, the equations are usually not the same as the ones the calculator has given them.) They then use the two equations found to give the amount of daylight for a particular day of the year and to compare their answers with the amounts given by the website.

Hand out the Finding Daylight BLMs. Assign each group one of the groups of cities. Each member of the group should have a different city. Be sure that the groups understand how to use the tables to find the amount of daylight.

Each of the cities will have different answers. In general, the farther north the city the more the amount of daylight will vary. Have the students show their calculator work. Both equations entered should do a very good job of covering the data points. The prediction of daylight for their cities should be very close to the amount of daylight found on the website. Have each student report his/her results to the class.

**Activity 8: Adding the Sine and Cosine Functions to the Library of Functions**

(GLEs: 4, 6, 8, 12, 13, 25, 28)

Materials List: Library of Functions – The Sine Function and The Cosine Function BLM, folder, paper, pencil

Each of the functions studied in this course will have an entry in the Library of Functions. The purpose of this ongoing activity is to prepare a portfolio of functions. It is important that a student have a visual picture of the function, its important characteristics, and what role it might play in real life. The sine and cosine functions should be completed in this unit. The student should begin with the parent functions: $f(x) = \sin x$ and $f(x) = \cos x$. In general, the following characteristics should be covered:

- domain and range
• local and global characteristics such as symmetry, continuity, whether the function has local maxima and minima with increasing/decreasing intervals or is a strictly increasing or strictly decreasing function with existence of an inverse, end-behavior, and periodicity.
• discuss the common characteristics as well as the differences of the two functions
• examples of translation, reflection, and dilation in the coordinate plane
• a real-life example of how the function family can be used showing the 4 representations of a function along with a table of select values
• describing the amplitude, period, phase shift, and change in the sinusoidal axis in the example

Hand out the Library of Functions – Sine Function and Cosine Function BLM to each student. This should be completed and added to their Library of Functions Portfolio.

Sample Assessments

General Assessments

• One or more writing assessments should be assigned for the unit. Students have added to their notebook glossaries throughout this unit. They have also had a short writing assignment with many of their activities. Therefore, one of the assessments should cover this material. Look for understanding of how the term or concept is used.

This unit introduced radians as a measure. Therefore a good question for the students would be: For what types of real-life problems should the angle measure be in radians rather than degrees? Justify your answer. A possible answer would be: Since radian measure is measuring the length of an arc, it is best used for time or distance problems. In most real-life situations, the independent variable of a periodic function involves time or distance. Angles are rarely mentioned.

• One of the favorite methods of review is a weekly “spiral”, a handout of 10 or so problems covering work previously taught in the course. Tie them to the study guide for a unit test or a midterm exam. Students should complete at least three “spirals” during this unit. The General Assessments Spiral BLM for this unit contains some problems that review algebra, and others that cover the trigonometric ratios introduced in geometry and reviewed in Unit 4 of this course. Another “spiral” should cover finding coterminal angles and reference angles.

• Students should also engage in a group activity that will be assessed. Assign each group a real-life situation that can be modeled by a periodic function. (One such activity is given.) See the list in Activity 7 for other suggestions. Students are responsible for finding the data and presenting their findings to the class.
The presentation should include
1. The estimated amplitude, period, phase shift, and vertical translation.
2. An algebraic representation that models the data.
3. Predictions based on the function found in #2.

The scoring rubric should include
- teacher observation of group interaction and work
- explanation of each group’s problem to class
- work handed in by each member of the group

Activity-Specific Assessments

- **Activity 1:** Students should demonstrate proficiency in finding exact values of angles in both radians and degrees, in drawing angles in standard position, and in finding coterminal angles.

- **Activity 3:** Students should demonstrate proficiency with questions such as the following:
  1. An arc with length 2.6 radians is measured from the end-point (1, 0) to P(x, y). Find the x- and y-coordinates.
  2. An arc has endpoints (1, 0) and \((-\frac{1}{2}, -\frac{\sqrt{3}}{2})\). What is the length of the arc?

- **Activity 8:** Students should turn in the pages they have completed for the Library of Functions on the sine and cosine functions.
  Scoring rubric should include the following
  1. Thorough coverage of the material.
  2. The material presented is accurate.
  3. The work is neat and organized. Descriptions are given in sentences.
  4. The graphs are labeled, drawn to scale and are correct.
Advanced Math – Pre-Calculus
Unit 6: Additional Topics in Trigonometry

Time Frame: 4.5 weeks

Unit Description

This unit continues the study of trigonometry looking now at the trigonometric identities, the inverse trigonometric functions, and trigonometric equations. The study of polar coordinates and complex numbers is also included in this unit. Teachers might also want to teach parametric equations as part of this unit. The activities for parametric equations are found in Unit 8.

Student Understandings

Students understand the inverse trigonometric functions. They can evaluate inverse relations and identify the principal values that make the inverse a function. They are able to evaluate a composition that involves the trigonometric functions. Students are introduced to the trigonometric identities and learn to use them to evaluate trigonometric functions, to simplify trigonometric expressions, and to solve trigonometric equations. Finally, students can work in both the complex number system and the polar system.

Guiding Questions

1. Can students identify the other four trigonometric functions, define them according to the unit circle, and sketch their graphs?
2. Can students write a trigonometric equation in the form of an inverse relation?
3. Can students find the values of the inverses of the trigonometric functions?
4. Can students find the principal values of inverse trigonometric functions?
5. Can students evaluate the compositions of trigonometric functions?
6. Can students use the sum and difference identities and the double angle identities?
7. Can students use the identities to solve trigonometric equations?
8. Can students graph polar coordinates and simple polar equations?
9. Can students change from rectangular coordinates to polar coordinates and vice versa?
10. Can students graph a complex number and find its absolute value?
11. Can students write a complex number in polar form?
12. Can students multiply and divide complex numbers written in polar form?
Unit 6 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number and Number Relations</strong></td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>Read, write, and perform basic operations on complex numbers (N-1-H) (N-5-H)</td>
</tr>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>Translate and show the relationship among the non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics for the function (A-3-H)</td>
</tr>
<tr>
<td>8.</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)</td>
</tr>
<tr>
<td><strong>Measurement</strong></td>
<td></td>
</tr>
<tr>
<td>11.</td>
<td>Calculate angles measures in degrees, minutes, and seconds (M-1-H)</td>
</tr>
<tr>
<td>12.</td>
<td>Explain the unit circle basis for radian measure and show its relationship to degree measure of angles (M-1-H)</td>
</tr>
<tr>
<td>13.</td>
<td>Identify and apply the unit circle definition to trigonometric functions and use this definition to solve real-life problems (M-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>16.</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>25.</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H)(P-5-H)</td>
</tr>
<tr>
<td>28.</td>
<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
</tr>
</tbody>
</table>

**Sample Activities**

This unit continues the study of the trigonometric functions. The four remaining trigonometric functions are introduced. Students are encouraged to tie those functions to the sine and cosine functions and graph them using translations and dilations. The fundamental identities are revisited. A sample spiral asks students to reduce expressions to a single trigonometric function. Inverse functions, the sum and difference formulas and the double angle formulas are introduced. Students learn to graph complex numbers, write them in polar form, then multiply and divide them. The unit ends with the graphs of polar functions.
Ongoing: Glossary notebook

Materials List: index cards 3 x 5 or 5 x 7, What Do You Know about These Topics in Trigonometry? BLM, pencil

Students continue the two methods used to help them understand the vocabulary for this course. As was done in the previous units begin by having each student complete a self-assessment of his/her knowledge of the terms for this unit using the modified vocabulary self awareness chart [view literacy strategy descriptions]. What Do You Know about These Topics in Trigonometry? BLM. Students should continue to make use of a modified form of vocabulary cards [view literacy strategy descriptions]. Add new cards for the following terms as they are encountered in the unit: inverse sine function, inverse cosine function, inverse tangent function, co-functions, even/odd identities, sum and difference identities, double angle identities, polar coordinate system, pole, polar axis, polar coordinates.

Activity 1: The Graphs of the Four Remaining Functions (GLEs: 4, 28)

Materials List: Graphs of the Four Remaining Functions BLM, graph paper, pencil, ruler

Students have already been introduced to the trigonometric functions \( y = \tan x \), \( y = \cot x \), \( y = \sec x \) and \( y = \csc x \) in Units 4 and 5. They were defined with the reciprocal, quotient, and Pythagorean identities in Unit 4 and in Activities 3 and 5 in Unit 5. In this activity, students look at the graphs of the tangent, cotangent, secant, and cosecant functions. Problems will have students graph the functions using translations and dilations. Students should be aware that while the sine and cosine functions are continuous, the remaining four are not. They should understand the asymptotic behavior of each of the remaining four. Use the five points: the maximum, minimum, and intercepts to graph.

For \( y = \csc x \): For each of the \( x \)-intercepts of \( y = \sin x \), \( \csc x \) is undefined. This will result in a vertical asymptote at each of the points. Notice that the vertical asymptotes occur at the beginning, middle, and end of the period.

\[
\csc x = \frac{1}{\sin x}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \sin x )</th>
<th>( \csc x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>( 3\pi/2 )</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>0</td>
<td>undefined</td>
</tr>
</tbody>
</table>

![Graph of the four remaining trigonometric functions](image)
For \( y = \sec x \): For each of the \( x \)-intercepts of \( y = \cos x \), \( \sec x \) is undefined. This will result in a vertical asymptote at each of those points. Note this occurs at \( \pi/2 \) and \( 3\pi/2 \).

\[
\sec x = \frac{1}{\cos x}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \cos x )</th>
<th>( \sec x )</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>( 2\pi )</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

For \( y = \tan x \): \( \tan x = \frac{\sin x}{\cos x} \)

- The period of the tangent function is \( \pi \) units.
- There are vertical asymptotes where \( \cos x = 0 \).
- Beginning at \( x = \pi/2 \), the asymptotes occur every \( \pi \) units.
- There is no amplitude since the graph is unbounded. The value \( a \) causes a vertical dilation.

For \( y = \cot x \): \( \cot x = \frac{\cos x}{\sin x} \)

- The period of the cotangent function is \( \pi \) units.
- There are vertical asymptotes where \( \sin x = 0 \).
- Beginning at the origin, the asymptotes occur every \( \pi \) units.
- The asymptotes of the graph of \( y = \tan x \) are the \( x \)-intercepts of the graph of \( y = \cot x \).
- The graphs of \( y = \tan x \) and \( y = \cot x \) have the same \( x \)-values for \( y \)-values of 1.
Be sure that students understand that the vertical lines that appear on the calculator graphs of the secant, cosecant, tangent and cotangent functions are the “place holders” for the vertical asymptotes. When students replicate the graph the vertical asymptotes should appear as dotted lines to show that they are merely guidelines. Note that neither the secant nor cosecant function has an amplitude since neither are bounded functions. However, the value $a$ in $y = acsc \ x$ or $y = asec \ x$ will affect the function since it will change the value of the maximum or minimum point.

Example: Sketch the graph of $y = 2sec \ (2x - 4)$ over two periods. Find the period and the phase shift. List the asymptotes. Find the location of the maximum and minimum points and their value.

The period is $\pi$.

$$\text{period} = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$$

The phase shift is 2 units to the right.

The vertical asymptotes will be found at $x = 2 + \frac{\pi}{4}$ and $2 + \frac{3\pi}{4}$. The value 2 in the equation $y = 2sec \ (2x - 4)$ will change the value of the maximum and minimum. The minimum value is 2 and the maximum value is -2. The minimum points are located at $x = 2, 2 + \pi$. and the maximum points are located at $2 + \frac{\pi}{2}$ and $2 + \frac{3\pi}{2}$.

Encourage the students to use their knowledge of the sine and cosine functions, their $x$-intercepts, and the maximum and minimum points to help graph the other four trigonometric functions. Hand out the Graphs of the Four Remaining Functions BLMs. Let the students work in their groups. Remind them that the axes are labeled and scaled properly. Vertical asymptotes should be dotted lines using rulers with the graphs carefully drawn.
### Activity 2: The Inverse Trigonometric Functions (GLEs: 4, 6, 8, 12, 13)

Materials List: Working with Inverse Trigonometric Functions BLM, calculator, pencil

Vocabulary to be added: inverse sine function, $\sin^{-1} x$, inverse cosine function, $\cos^{-1} x$, inverse tangent function, $\tan^{-1} x$

The sine and cosine functions are periodic functions meaning they repeat their values every $2\pi$ units. It is this attribute that prevents the sine and cosine, in fact all of the trigonometric functions, from having an inverse that is also a function. Suppose $\sin x = .556$. We need to know the value of $x$. There are two answers in the fundamental period of sine and an infinite number over the domain of sine. Real life applications usually require a unique answer. Therefore, it is necessary to restrict the domain with sine (in fact with all of the trigonometric functions) so that the domain and range values are paired in a one-to-one manner.

A good example is a calculator. Given the equation $\sin x = \frac{1}{2}$, what value should the calculator display for $x$? Have the students set their calculators to degrees and work each of the following:

1) $\sin x = \frac{1}{2}$  
2) $\sin x = -\frac{1}{2}$  
3) $\cos x = \frac{1}{2}$  
4) $\cos x = -\frac{1}{2}$  
5) $\tan x = 1$  
6) $\tan x = -1$

What is the value of $x$ for each one?

1) $30^\circ$, 2) $-30^\circ$, 3) $60^\circ$, 4) $120^\circ$ 5) $45^\circ$, 6) $-45^\circ$

Present the following table:

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Principal Values</th>
</tr>
</thead>
</table>
| $y = \sin^{-1} x$  
$x = \sin y$ | $-1 \leq x \leq 1$ | $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$  
$-90^\circ \leq \sin^{-1} x \leq 90^\circ$ |
| $y = \cos^{-1} x$  
$x = \cos y$ | $-1 \leq x \leq 1$ | $0 \leq \cos^{-1} x \leq \pi$  
$0^\circ \leq \cos^{-1} x \leq 180^\circ$ |
| $y = \tan^{-1} x$  
$x = \tan y$ | $-\infty < x < \infty$ | $-\frac{\pi}{2} \leq \tan^{-1} x \leq \frac{\pi}{2}$  
$-90^\circ \leq \tan^{-1} x \leq 90^\circ$ |
| $y = \cot^{-1} x$  
$x = \cot y$ | $-\infty < x < \infty$ | $0 \leq \cot^{-1} x \leq \pi$  
$0^\circ \leq \cot^{-1} x \leq 180^\circ$ |
| $y = \sec^{-1} x$  
$x = \sec y$ | $x \leq -1$ or $x \geq 1$ | $0 \leq \sec^{-1} x \leq \pi$, $\sec^{-1} x \neq \frac{\pi}{2}$  
$0^\circ \leq \sec^{-1} x \leq 180^\circ$, $\sec^{-1} x \neq 90^\circ$ |
| $y = \csc^{-1} x$  
$x = \csc y$ | $x \leq -1$ or $x \geq 1$ | $-\frac{\pi}{2} \leq \csc^{-1} x \leq \frac{\pi}{2}$  
$-90^\circ \leq \csc^{-1} x \leq 90^\circ$, $\csc^{-1} x \neq 0$ |
The notation for the principal value of an inverse trigonometric function is either $\sin^{-1}$ or $\text{Arcsin}$. This unit uses the $\sin^{-1}$ notation. Remind students that this does not refer to the reciprocal of the function.

Use the graphing calculator to introduce the graphs of each of the inverse functions in the table on the previous page. Use the trace feature to show that the domains of $y = \sin^{-1}x$, $y = \cos^{-1}x$, $y = \sec^{-1}x$, and $y = \csc^{-1}x$ are restricted.

Below are some examples of problems students will encounter on the Working With Inverse Trigonometric Functions BLM.

Example 1: Evaluate. Give the exact answer in radians: $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Solution:

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$. This is a unit circle value. The domain of the Principal value is $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$.

Example 2: Find the exact value of $\sin(\cos^{-1}\frac{3}{5})$

Let $\theta = \cos^{-1}\frac{3}{5}$. Set up a right triangle with the side adjacent to $\theta$ equal to 3 and the hypotenuse equal to 5. Use the Pythagorean Theorem to find the side opposite. Then $\sin(\cos^{-1}\frac{3}{5}) = \sin \theta$ and $\sin \theta = \frac{4}{5}$.

Example 3: Find $\cos(\sin^{-1}x)$. Follow the same procedure as shown in example 2.

Let $\theta = \sin^{-1}\frac{x}{1}$. Set up a right triangle with the side opposite to $\theta$ equal to $x$ and the hypotenuse equal to 1. Use the Pythagorean Theorem to find the side adjacent. Then $\cos(\sin^{-1}x) = \cos \theta$ and $\cos \theta = \sqrt{1-x^2}$. Therefore $\cos(\sin^{-1}x) = \sqrt{1-x^2}$.

Have students use their calculators to graph $\cos(\sin^{-1}x)$ and $\sqrt{1-x^2}$ on the same screen to verify that the two are equal.

Hand out the Working with Inverse Trigonometric Functions BLMs. Each student should work individually. Put students into groups to check their answers.
Activity 3: Solving Trigonometric Equations Over Several Periods (GLEs: 4, 6, 11, 25)

Materials List: Solving Trigonometric Equations BLM, paper, pencil, graph paper, graphing calculator

This activity is designed to give students practice in solving trigonometric equations both algebraically and graphically. Connect solving equations to finding the zeros of functions by reminding them that another way of saying “solve $3\cos(2x) = 2,$” is to ask them to find the zeros of $f(x) = 3\cos(2x) - 2$ in the interval $[0, 360^\circ).$ This activity will emphasize the multiple answers obtained due to the periodic nature of the function. Prior to the activity you might want to have students solve problems such as $\cos x = 0.56$ for $0 \leq x < 360^\circ$ illustrating the answer with a graph. Each problem should be worked as follows:

Example 1:

$\text{Solve } 3\cos(2x) = 2, \text{ for } 0 \leq x < 360^\circ$

a) Solve algebraically:

$\cos(2x) = \frac{2}{3}$

$\cos^{-1}\left(\frac{2}{3}\right) = 2x$

$48.2^\circ = 2x$

$x = 24.1^\circ$

*The period for $\cos 2x$ is $180^\circ,$ so there are 4 answers $24.1^\circ,$ $155.9^\circ,$ and $24.1^\circ + 180$ or $204.1^\circ$ and $155.9^\circ + 180$ or $335.9^\circ$*

Solve graphically:

$y1: \cos 2x$

$y2: 2/3$

*Set the window $[0, 360]$ by $[-2, 2]$

*Use the CALC 5:intersect feature of the graphing calculator to obtain the 4 values.*

Example 2:

$\text{Solve } 2\sin^2 x - \sin x - 1 = 0 \text{ for } 0 \leq x < 360^\circ$

(a) Solve algebraically:

$(2\sin x + 1)(\sin x - 1) = 0$

$2\sin x + 1 = 0$ or $\sin x - 1 = 0$

$\sin x = -\frac{1}{2} \text{ or } \sin x = 1$

$x = 210^\circ, 330^\circ, 90^\circ$

Solve graphically:

*Use Calc 2:zero to find each of the x-intercepts.*
Hand out the Solving Trigonometric Equations BLMs. Remind the students that the algebraic solution requires that work must be shown. Bald answers (giving answers only) are not acceptable. Let the students work in their groups.

**Activity 4: Using the Fundamental Identities to Solve Trigonometric Equations**

(GLEs: 6, 11, 25)

Materials List: Using the Fundamental Identities to Solve Trigonometric Equations BLM, pencil, paper, graphing calculators

Students were introduced to the fundamental identities in Unit 4, Activity 1. They used the identities to evaluate a function, simplify a trigonometric expression, and verify identities. The spiral for this unit will review those identities. Be sure that students have completed it prior to working this set of problems. Students should solve each equation algebraically using the identities and verify their answers graphically.

**Example 1:** Solve $2\sin^2 x = 1 + \cos x$ for $0 \leq x < 360^\circ$

\begin{align*}
a) \text{ Set the equation equal to zero } & 2\sin^2 x - 1 - \cos x = 0 \\
b) \text{ Replace } \sin^2 x \text{ with } 1 - \cos^2 x & 2(1 - \cos^2 x) - 1 - \cos x = 0 \\
c) \text{ Solve } f(x) = 0 \text{ for } 0^\circ \leq x < 360^\circ & -2 \cos^2 x - \cos x + 1 = 0 \\
d) \text{ Multiply through by } -1 \text{ and factor.} & 2 \cos^2 x + \cos x - 1 = 0 \\
& (2\cos x - 1)(\cos x + 1) = 0 \\
& \cos x = \frac{1}{2} \text{ or } \cos x = -1 \\
& x = 30^\circ, 150^\circ, 180^\circ
\end{align*}

*The graph of Y1: $2\sin^2 x$ and Y2: $1 + \cos x$ showing the solution of $180^\circ$*

Hand out Using the Fundamental Identities to Solve Trigonometric Equations BLM. Students should work in their groups to solve each of the problems.
Activity 5: Properties and Formulas for the Trigonometric Functions (GLEs: 4, 12, 25)

Materials List: Working with the Properties and Formulas for the Trigonometric Functions BLM, pencil, calculator

In this activity students are introduced to the composite argument properties involving (A + B) and (A – B), the odd-even properties, the co-function properties, and the double angle properties for sine, cosine, and tangent. Go over each of the following with the class.

The Odd-Even Properties
\[ \sin(-x) = -\sin x \quad \text{(odd function)} \]
\[ \cos(-x) = \cos x \quad \text{(even function)} \]
\[ \tan(-x) = -\tan x \quad \text{(odd function)} \]

The Cofunction Properties: Functions of \((90^\circ - \theta)\) or \(\left(\frac{\pi}{2} - x\right)\)
- The cosine of an angle equals the sine of the complement of that angle.
- The cotangent of an angle equals the tangent of the complement of that angle.
- The cosecant of an angle equals the secant of the complement of that angle.

Sum and Difference Formulas
\[ \cos(A + B) = \cos A \cos B - \sin A \sin B \]
\[ \cos(A - B) = \cos A \cos B + \sin A \sin B \]
\[ \sin(A + B) = \sin A \cos B + \cos A \sin B \]
\[ \sin(A - B) = \sin A \cos B - \cos A \sin B \]
\[ \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \]
\[ \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \]

Double Angle Formulas
\[ \sin 2x = 2 \sin x \cos x \]
\[ \tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \]
\[ \cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \]

The odd and even function properties could also be introduced when studying the graphs of the trigonometric functions. Tie the co-function properties to the right triangle ratios.
Example 1: Solve over $0 \leq x < 2\pi$, $4 \sin x \cos x = \sqrt{3}$.

Use the double angle identity for sine.

$2(2 \sin x \cos x) = \sqrt{3}$

$2 \sin x \cos x = \frac{\sqrt{3}}{2}$

$\sin 2x = \frac{\sqrt{3}}{2}$

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 2x$

$2x = \frac{\pi}{3}, \frac{2\pi}{3}$

$x = \frac{\pi}{6}, \frac{\pi}{3}$

The period of $\sin 2x$ is $\pi$. Therefore, there are two more values in $0 \leq x < 2\pi$.

The other two solutions are $\frac{\pi}{6} + \frac{7\pi}{6}$ and $\frac{\pi}{3} + \frac{4\pi}{3}$

Example 2: Solve $2 \cos 3x \cos x - 2 \sin 3x \sin x = \sqrt{3}$ over $0 \leq x < 2\pi$

$2(\cos 3x \cos x - \sin 3x \sin x) = \sqrt{3}$

$2 \cos 4x = \sqrt{3}$

$\cos 4x = \frac{\sqrt{3}}{2}$

$4x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$4x = \frac{\pi}{6}, \frac{11\pi}{6}$

$x = \frac{\pi}{24}, \frac{11\pi}{24}$ The period of $\cos 4x$ is $\pi$. There are six more answers:

$\frac{13\pi}{24}, \frac{23\pi}{24}, \frac{25\pi}{24}, \frac{35\pi}{24}, \frac{37\pi}{24}, \frac{47\pi}{24}$

Hand out Working with the Properties and Formulas for the Trigonometric Functions BLM. Let the students work with partners or with their groups on this set of problems. Remind them to give exact answers where possible in solving the equations in #2. (Only (2a) and (2b) do not have exact answers.)
Activity 6: Playing Mr. Professor (GLEs: 4, 6, 8, 11, 12, 13)

Materials List: pencil, paper, chalk board, transparencies, or chart paper, calculator, Answers to Problem Set for Playing Mr. Professor BLM

In this activity, students will gain some additional practice in graphing the trigonometric functions. They will use the sum and difference and the double angle formulas, work with the inverse functions, and apply the odd-even and co-function properties. They will become experts using the game Mr. Professor.

Introduce the professor know–it–all strategy (view literacy strategy descriptions). Divide the students into groups. Explain to the students that each group will be called upon to become a team of math prodigies. They will have a chance to demonstrate their expertise in Part I: graphing, Part II: solving equations, Part III: using the identities and formulas to simplify expressions, and Part IV: finding exact values using inverse trigonometric functions, by answering the questions posed by the rest of the class. The other students may challenge any of the answers given. The team may confer on each question but each member of the group should have a chance to explain. There are 15 problems, more than can be covered in one period. Choose those that will provide students additional needed work. An example problem should be used to remind students how the group should respond to their peers’ questions. Answers to the problem set for Playing Mr. Professor BLM has been provided. The questions should cover the points below.

- For the graphs
  - the graph of the parent function, its period, x- and y-intercepts, location of asymptotes, and the location and value of the maximum and minimum
  - the period of the function
  - the phase shift
  - the location of asymptotes
  - the maximum and minimum values and their locations
  - the x-and y-intercepts

Once all questions have been answered, the team should sketch the graph with the given information. Students may then check by graphing the function using calculators.

- Problems for the graphing portion of the game
  1. \( y = -\tan\left( x + \frac{\pi}{4} \right) \)
  2. \( y = 3\sec\left( x + \frac{\pi}{4} \right) - 1 \)
  3. \( y = \cot\left( 2x + \frac{\pi}{3} \right) \)
  4. \( y = 4\csc(2x - 3\pi) + 2 \)
Questions for solving equations
- What functions are in the equation?
- What identities or properties can be used to yield an equivalent equation involving only one function?
- What operations must be used to solve the equation?
- Does the graphical solution agree with the algebraic solution?

Problems for solving equations. Solve over $0 \leq x < 360^\circ$.
1. $1 + \cos x = 4 \sin^2 x$
2. $3 \sin x + 2 = \cos 2x$
3. $\cos 2x \cos x + \sin 2x \sin x = -\frac{1}{2}$
4. $\frac{\tan 3x - \tan x}{1 + \tan 3x \tan x} = \sqrt{3}$

Questions for simplifying an expression into a single trigonometric function.
- What functions should be replaced?
- What identity should be used?
- What operations can be used?

Problems for simplifying expressions
1. $\frac{\cos x}{1 + \sin x} + \tan x$
2. $\sec x + \csc x$
3. $1 + \tan x$
3. $\sin x (\cos x + \sin x \tan x)$
4. $\frac{\sec x - \cos x}{\sin^2 x \sec^2 x}$

Questions for finding exact values using inverse trigonometric functions
Each of the problems involves either the sum, difference, or the double angle formulas.
- What right triangles must be drawn?
- Which quadrant is the right triangle drawn?
- Which formula should be used?
- Why is it necessary to use the principal values for the inverse functions?
Problems to be used

Find the exact values for each of the following:

1. \( \sin \left( \tan^{-1} 3 - \cos^{-1} \frac{1}{3} \right) \)

2. \( \cos \left( \cos^{-1} \frac{1}{4} + \cos^{-1} \left( -\frac{1}{3} \right) \right) \)

3. \( \sin \left( \cos^{-1} \frac{1}{3} + \tan^{-1} (-2) \right) \)

4. \( \tan 2 \left( \sin^{-1} x \right) \)

Activity 7: Polar Representation of Complex Numbers (GLEs: 1, 12, 16)

Materials List: Polar Representation of Complex Numbers BLM, graph paper, pencil

Vocabulary to cover: polar coordinate system

Students have had an introduction to complex numbers in Unit 4 of the Algebra II course but will probably need to review operations with complex numbers. Specifically cover

- the definition of a complex number (Stress to the students that the \( i \) denotes the imaginary part of the number and is not included as part of the number.)
- the complex conjugate of \( a + bi \) is \( a - bi \)
- the operations of addition, subtraction, multiplication, and division

Since complex numbers are ordered pairs of real numbers, it is possible to represent them graphically by points in a Cartesian Coordinate System called a complex plane. In the complex plane, the horizontal axis represents the real part of the complex number \( a + bi \) and the vertical axis is the imaginary part. The number can be represented as a point in the plane or as an arrow from the origin to the point. The length of the arrow is \( \sqrt{a^2 + b^2} \). It is called the absolute value of a complex number. Students should see the connection between graphing a complex number and working with vectors.

To work effectively with the powers and roots of complex numbers, it is helpful to write them in polar form. Use the diagram below to illustrate the definition of the polar form of a complex number:

\[ z = r(\cos \theta + i \sin \theta) \] where \( r \) is the magnitude of \( z \) and \( \theta \) is called the argument of \( z \) (either in degrees or radians).
The expression \( r \cos \theta + i \sin \theta \) is written "cis\( \theta \)" where \( c \) comes from cosine, \( i \) from the unit imaginary number, and \( s \) from sine. With this information it is possible to transform complex numbers in Cartesian form into polar form using the following formulas:

\[
r = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan \theta = \frac{b}{a}.
\]

Example: Transform the complex number \( z = 3 - 4i \) to polar form.

Sketch in the complex plane.

\[
r = \sqrt{3^2 + (-4)^2} = 5
\]

\[
\theta = \tan^{-1}\left(\frac{-4}{3}\right)
\]

\[
\theta = 306.869\ldots \approx 307^\circ \text{ or } 5.4 \text{ radians}
\]

\[
z = 5 \text{cis} 307^\circ
\]
Cover the following properties with the students:

- **Product of Two Complex Numbers in Polar Form:** \( z_1z_2 = r_1r_2\text{cis}(\theta_1 + \theta_2) \)
- **Reciprocal of a Complex Number in Polar Form:** \( \frac{1}{z} = \frac{1}{r}\text{cis}(-\theta) \)
- **Quotient of Two Complex Numbers in Polar Form:** \( \frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2) \)

Hand out the Polar Representation of Complex Numbers BLMs. Students should work individually on the handout then check their answers with their groups.

**Activity 8: The Graphs of Polar Functions (GLEs: 4, 16, 25)**

Materials List: The Graphs of Polar Functions BLM, graphing calculator, pencil

Vocabulary to cover: pole, polar axis, polar coordinates

Cover the polar coordinate system and the coordinate changes needed to go from polar to rectangular and from rectangular to polar. Students will need some practice in plotting polar coordinates as well as converting from one system to the other.

Most advanced math textbooks have a section on graphs of polar functions. Usually they are limited to finding tables of values for the various equations and to hand-plotting using polar graph paper. Students learn the names of some of the graphs of the more famous polar functions. This activity will utilize a modified word grid (view literacy strategy descriptions) to give students a chance to study the makeup of one of those graphs whose equations are of the form \( f(\theta) = a\sin(n\theta) \) or \( f(\theta) = a\cos(n\theta) \) where \( n > 1 \) and \( n \) is an integer. It is called a rose curve. Students will use the graphing calculators for this activity. Have students go to MODE and set the graphing calculators to polar (pol).

Remember that the range of \( \sin \theta \) is \( \{y: -1 \leq y \leq 1\} \) so the value of \( a \) in the two equations determines the size of the graph. To obtain an accurate picture the screen needs to be “square.” To begin, let \( a = 1 \), set the x and y values in the window to be [-1, 1], and then press ZOOM 5 to “square” the graph.

Investigate

1) Symmetry -
   - with respect to the polar axis (in the rectangular system, the horizontal axis)
   - with respect to the pole (in the rectangular system, the origin (0, 0))
   - with respect to the line \( \theta = \frac{\pi}{2} \) (in the rectangular system, the vertical axis)

2) Zeros – When is \( r = 0? \) On the graph when does the graph return to the pole?
3) Domain – To begin we use \([0, 2\pi]\). What is the least interval of that domain that gives a complete graph?

Students should investigate each of the rose curves filling out the grids with the information they obtain. To answer some of the questions it is helpful to either trace the graph or look at the table values. Try setting the \(\Delta Tbl\) to \(\frac{\pi}{24}\). This is the same as the \(\theta\) step in the window. Students can scroll through the table to find the answers. By filling in the grids, students investigate each polar equation symbolically, numerically, and graphically. Students will give verbal descriptions of what is happening when answering the questions that follow the grid.

Distribute The Graphs of Polar Functions BLMs. Have the students complete the grids working alone. Once all have finished, call on students to give the answers. Students should then complete the writing activity. A discussion of what each has written should end the session. Allow time for students to quiz each other over the content of the modified word grids in preparation for tests and other class assignments.

Students enjoy creating designs using the various rose curves. If you have the equipment to download the picture from the graphing calculator to a computer, students can then make a permanent copy. The designs can also be put into a class PowerPoint presentation.

**Sample Assessments**

**General Assessments**

- A writing assessment should be assigned for the unit. A good writing assessment for this unit is to have a student explain how to solve a trig equation algebraically. “Your neighbor was absent from school when your class learned how to solve \(3\sin x - 2\cos^2 x = 0\) over \(0 \leq x < 360\). Explain to her how this problem is to be solved.”
- Review of previously learned concepts should be ongoing throughout the unit. One of the favorite methods is a weekly “spiral”, a handout of 10 or so problems covering work previously taught in the course. Tie them to the study guide for a unit test or a midterm exam. There should be at least three spirals for this unit. The Spiral BLM gives students more practice in using the fundamental identities to simplify a trigonometric expression. Another spiral should cover the algebra of complex numbers learned in Algebra II since this is a prerequisite for the polar form of complex numbers. The third spiral should look again at the material in Units 1, 2, and 3.
- Students will demonstrate proficiency in working with equations, identities and the formulas without benefit of calculator.
Activity-Specific Assessments

- **Activity 2**: Students should demonstrate proficiency in working with the domain and principal values of the inverse trigonometric functions.

- **Activities 3 and 4**: Students should demonstrate proficiency in solving equations both graphically and algebraically.

- **Activity 7**: Students should demonstrate proficiency in moving between the rectangular coordinate system and the polar coordinate system. They should be able to name more than one set of coordinates for a point in the polar coordinate system.
Advanced Math – Pre-Calculus
Unit 7: Sequences and Series

Time Frame: 2.5 weeks

Unit Description

This unit introduces finite and infinite sequences and series. A sequence can be thought of as a function with the inputs being the natural numbers. As a result the four representations of functions apply. The unit covers the sums of finite series and infinite series.

Student Understandings

Students will be able to find the terms of a sequence given the \( n \)th term formula for that sequence. They can recognize an arithmetic or geometric sequence, find the explicit or recursive formula for that sequence, and graph the sequence. With infinite sequences they will find limits if they exist. Students can expand a series, written in summation notation, and find the sum. They can use the formulas for the sum of a finite arithmetic or geometric series to find the sum of \( n \) terms. They are able to tell whether or not an infinite geometric series has a sum and find the sum if it does exist. They are able to model and solve real-life problems using sequences and series.

Guiding Questions

1. Can students recognize a sequence as a function whose domain is the set of natural numbers?
2. Can students graph a sequence?
3. Can students recognize, write, and find the \( n \)th term of a finite arithmetic or geometric sequence?
4. Can students give the recursive definition for a sequence?
5. Can students recognize, write, and find the sum of an arithmetic or geometric series?
6. Can students determine if the sum of an infinite geometric series exists and, if so, find the sum?
7. Can students recognize the convergence or divergence of a sequence?
8. Can students find the limit of terms of an infinite sequence?
9. Can students use summation notation to write sums of sequences?
10. Can students use sequences and series to model and solve real-life problems?
Unit 7 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>6</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics of the function</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>26</td>
<td>Represent and solve problems involving nth terms and sums for arithmetic and geometric series (P-2-H)</td>
</tr>
</tbody>
</table>

Sample Activities

**Ongoing: Glossary**

Materials List: index cards, What Do You Know About Sequences and Series BLM, pencil

Students continue the glossary activity in this unit. They will repeat the two methods used in the previous units to help them understand the vocabulary for unit 7. Begin by having each student complete a self-assessment of his/her knowledge of the terms using a modified **vocabulary self awareness** chart (view literacy strategy descriptions) What Do You Know about Sequences and Series BLM. Do not give the students definitions or examples at this stage. Ask the students to rate their understanding of each term with a “+” (understand well), a “?” (limited understanding or unsure) or a “−” (don’t know). Over the course of the unit, students are to return to the chart, add information or examples, and re-evaluate their understandings of the terms or concepts. Students should continue to add to the **vocabulary cards** (view literacy strategy descriptions) introduced in Unit 1. Make sure that the students are staying current with the **vocabulary cards**. Time should be given at the beginning of each activity for students to bring their cards up to date.

Terms to add to the vocabulary list: finite sequence, infinite sequence, terms of a sequence, arithmetic sequence, common difference, recursion formula, geometric sequence, common ratio, finite series, infinite series, nth partial sum, convergence, divergence, summation (sigma) notation, index of summation, lower limit of summation, upper limit of summation.
Activity 1: Arithmetic and Geometric Sequences (GLEs: 4, 24, 26)

Materials List: Arithmetic and Geometric Sequences BLM, graphing calculator, graph paper, pencil

Students need to be familiar with the following vocabulary for this activity: finite sequence, infinite sequence, terms of a sequence, nth term or explicit formula for a sequence, arithmetic sequence, common difference, geometric sequence, common ratio

A sequence is a function whose domain is the set of positive integers. It is a discrete function. Illustrate this with the graphs of \( f(x) = 1 + x, x > 0 \) and \( f(n) = 1 + n \) using the graphing calculator.

![Graph of y = 1 + x, x > 0](image1)

Point out that a sequence is usually represented by listing its values in order. For example, the sequence shown above would be written numerically as 2, 3, 4, 5, ... and algebraically with the rule for the nth term \( t_n = n + 1 \). (Many textbooks will use \( a \) in place of \( t \).) This is called the explicit formula.

Make the connection between arithmetic sequences and linear functions. The common difference in the formula for the arithmetic sequence is the slope in the linear function. The geometric sequence is an exponential function where the domain is the set of natural numbers. The common ratio was called the growth/decay factor in unit 3. As students work the problems that require them to find the nth term (explicit) formula, have them also draw a graph first of the sequence and then of the explicit formula.

Students should verify their graphs and formula using the graphing calculator. Put the calculator into sequence mode and graph each of the sequences. Students can turn on TRACE and see each term of the sequence. They can also use the table function to verify their formula.

Directions for graphing sequence above on the TI-83 calculator:
- ✓ Put the calculator into sequence mode
- ✓ Use Y=, enter the formula for the sequence
- ✓ Enter 1 in nMin
- ✓ Enter 2 in u(nMin). The value of the sequence at \( n = 1 \).
Set the window as follows, then use either TRACE or the TABLE function to find the terms:

Hand out the Arithmetic and Geometric Sequences BLMs. Students will identify arithmetic and geometric sequences, write nth term or explicit formulas given a sequence, and solve some real-life problems using arithmetic and geometric sequences. Have them work on this assignment individually, then check answers with their groups.

**Activity 2: Recursive Definitions (GLEs: 4, 24, 26)**

Materials List: Using the Recursion Formula BLM, graphing calculator, graph paper, pencil

Students need to be familiar with the following vocabulary for this activity: recursion formula

In addition to the explicit or general term formulas, there are also recursion formulas for sequences. A recursion formula is one in which a sequence is defined by giving the value of \( t_n \) in terms of the preceding term, \( t_{n-1} \).

Example:
A sequence is defined by the following formulas:

\[ t_1 = 3 \]
\[ t_n = 2t_{n-1} + 5 \]

The second formula above states that the nth term is 5 more than twice the previous term. The sequence begins with 3. Each of the following terms is found using \( t_n = 2t_{n-1} + 5 \):
The sequence 3, 11, 27, 59,… is defined recursively by the recursion formula.

\[ t_1 = 3 \]

\[ t_n = 2t_{n-1} + 5 \]

Each recursion formula consists of two parts:
1) An initial condition that gives the first term of the sequence.
2) A recursion formula that tells how any term of the sequence is related to the preceding term.

A recursion formula may also be graphed. Press \( Y= \) and enter the recursion formula as shown below:

\[ \text{Plot1, Plot2, Plot3} \]

\[ n\text{Min}=1 \]

\[ u(n)=2u(n-1)+5 \]

\[ u(n)\text{Min}=3 \]

Use either TRACE or the TABLE function to find the terms.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( u(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>59</td>
</tr>
</tbody>
</table>

Have students give examples of the recursive and \( n \)th term formulas for the arithmetic and geometric sequences. They should see that the recursion formula for the arithmetic sequence adds the common difference while the recursion formula for the geometric sequence multiplies by the common ratio.

Look at some real-life examples of recursion formulas.

Example 1:

The population of a country in the southern hemisphere is growing because of two conditions
- the growth rate in the country is increasing at a rate of 2% per year
each year the country gains 30,000 immigrants

The recursion formula would be: \( t_1 = 6,000,000, \ t_n = 1.02t_{n-1} + 30,000 \). If the population is 6 million people, what will be the population each year for the next five years?

Example 2:

A trip to Cancun for the senior trip will cost $500 and full payment is due March 2nd. On September 1st, a student deposits $100 in a savings account that pays 5% per year, compounded monthly, and adds $50 to the account on the first of each month.

a) Find a recursive sequence that explains how much is in the account after \( n \) months.

\[
A_1 = 100, A_n = \left(1 + \frac{r}{n}\right)A_{n-1} + P
\]

\[
A_1 = 100, A_n = \left(1 + \frac{0.05}{12}\right)A_{n-1} + P
\]

b) List the amounts in the account for the first 6 months.

<table>
<thead>
<tr>
<th>September</th>
<th>October</th>
<th>November</th>
<th>December</th>
<th>January</th>
<th>February</th>
<th>March 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>150.50</td>
<td>201.25</td>
<td>252.26</td>
<td>303.52</td>
<td>355.04</td>
<td>406.81</td>
</tr>
</tbody>
</table>

c) How much would he have if he added $70 to the account?

$528.32 after he added to the account on March 1.

Put students into groups of three and distribute the Using the Recursion Formula BLMs. Students begin with problems that require them to write recursion and nth term formulas for sequences. They then develop real world problems using a math story chain (view literacy strategy descriptions). Story chains are very useful in teaching math concepts, while at the same time promoting reading and writing. How well students understand the concepts with which they have been working is reflected in the story problems they write and solve.

In a story chain the first student initiates the story. The next student adds a sentence and passes it to the third student to do the same. If a group member disagrees with any of the previous sentences, the group discusses the work that has already been done. Its members either agree to revise the problem or to move on as it is written. Once the problem has been written at least three questions should be generated. The group then works out a key and challenges another group to solve the problem.

<table>
<thead>
<tr>
<th>End of year 1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>6,150,000</td>
<td>6,303,000</td>
<td>6,459,060</td>
<td>6,618,241</td>
<td>6,618,241</td>
</tr>
</tbody>
</table>

The recursive definition works well on problems in which new amounts are added on a regular basis. Some ideas for problems are shown below:

- Saving money using compound interest
• Paying credit card debt
• Depreciation
• Population growth

Model the process for the students before they begin the Using the Recursion Formula BLM.

Student 1: Sue has a credit card balance of $3500 on her Master Card.
Student 2: She plans to pay $100 a month towards the balance.
Student 3: Her card charges 1% per month on any unpaid balance.

Some questions that could be asked include:

a) Find a recursion formula that represents the balance after making the $100 payment each month.
b) Using a graphing calculator, determine when Sue’s balance will be below $2000.
c) How many payments have been made?
d) How many months will it take to pay off the credit card debt?

Solution:
a) \( a_n = 3500, 1.01a_{n-1} - 100 \)
b) Enter the recursion formula in Y= and use a table to determine the number of months.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( S_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>10761.2</td>
</tr>
<tr>
<td>18</td>
<td>10761.2</td>
</tr>
<tr>
<td>19</td>
<td>10910.6</td>
</tr>
<tr>
<td>20</td>
<td>10910.6</td>
</tr>
<tr>
<td>21</td>
<td>11098.1</td>
</tr>
<tr>
<td>22</td>
<td>11098.1</td>
</tr>
</tbody>
</table>

\[ S_n = 1.01S_{n-1} - 100 \]

22 months $1989.50

c) 21 payments

d) 44 months with the last payment of $29.14

Activity 3: Series and Partial Sums (GLEs: 4, 24, 26)

Materials List: Series and Partial Sums BLM, graphing calculator, pencil, paper

Students need to be familiar with the following vocabulary for this activity: finite series, nth partial sum

If \( S_n \) represents the sum of \( n \) terms of a series, then \( S_n \) can be expressed explicitly or recursively as follows:

- Explicit definition of \( S_n \): \( S_n = t_1 + t_2 + t_3 + \ldots + t_n \)
- Recursive definition of \( S_n \): 
\[ S_0 = 0 \\
S_n = S_{n-1} + t_n \text{ for } n \geq 1 \]

Example: Find the sum of the cubes of the first 12 positive integers.
\[ 1^3 + 2^3 + 3^3 + \ldots + 12^3 \]

The TI-83 calculator can help to find the solution.

1. Press LIST
2. Right arrow to MATH and choose 5:sum( and press ENTER
3. Press LIST
4. Right arrow to OPS and choose 5: seq( and press ENTER

See the home screen below:

```
sum(seq(n^3, n, 1, 12, 1) = 6084
```

Either the calculator or the algebraic formulas can be used to find sums of arithmetic and geometric series.

Example of an arithmetic series:
Find the sum of the first 25 terms of 11 + 14 + 17 + 20 +…

a) Use the formula for the sum of the first \( n \) terms of an arithmetic series:
\[ S_n = \frac{n(t_1 + t_n)}{2} \]

Find \( t_{25} \):
\[ t_{25} = 11 + (25 - 1)3 \]
\[ = 83 \]

Fill in the formula:
\[ \frac{25(11 + 83)}{2} = 1175 \]

b) Using the calculator for the sum of the first \( n \) terms of an arithmetic series:
The series is arithmetic so the formula for its sequence is linear. The common difference is 3 (the slope) and the y-intercept is 8 (subtract 3 from the first term).
Using \texttt{sum(seq)) as shown below:

\[
\text{sum(seq(3n+8,n,1),25,1)} \\\ 1175
\]

Example of the sum of a geometric series: Find the sum of the first 10 terms of the geometric series 2 – 6 + 18 – 54+…

a) Use the formula for the sum of the first \(n\) terms of a geometric series using 
\[
S_n = \frac{a_1(1 - r^n)}{1 - r}, r \neq 1
\]

\textit{Solution: the sum} -29,524.

b) Using the calculator: The formula for the \(n\)th term of a geometric sequence is 
\[a_1r^{n-1}\]. In this series \(r = -3\) and \(a = 2\). Using \texttt{sum(seq)) put in the following:

\[
\text{sum(seq(2(-3)^((n-1)),n,1,18,13)}, -29524
\]

Hand out the Series and Partial Sums BLMs. Let the students work together in groups. Problem 6 is very much like the problem worked in Unit 3 Activity 3, “Saving for Retirement.” This is a good time to remind students that the problems involving saving or retiring debt are discrete problems, and that a geometric series can be used instead of the present or future value formulas.

**Activity 4: Infinite Sequences and Series (GLEs: 4, 6)**


Students need to be familiar with the following vocabulary for this activity: convergence, divergence

Students are usually introduced to limits in the Sequences and Series Unit. This activity is designed to first give students a visual picture of convergence and divergence of graphs of sequences before learning to find the limits by working with the symbolic formulas. Connect this activity to the end behavior of the functions previously studied. Begin with a discussion of convergence and divergence using the graphs below.
A modified opinionnaire (view literacy strategy descriptions) will now be utilized to connect the end-behaviors of functions already studied with the convergence or divergence of sequences whose nth term formulas are among those same families of functions. The student will be given nth term formulas and asked to predict the convergence or divergence of the sequence. Hand out Infinite Sequences and Convergence Part I BLMs and have the students complete the opinionnaire individually. When the students have finished, discuss each statement and the lessons learned. Once all questions have been answered, distribute the Infinite Sequences and Convergence Part II BLMs. Students should graph each function in sequence mode, sketch the graphs using

**Convergent Sequences**

![Graph of a convergent sequence](image)

- The values get closer and closer to a fixed value. There is a horizontal asymptote.

**Divergent Sequences**

![Graph of a divergent sequence](image)

- The sequence diverges to $+\infty$. The values grow in size becoming infinitely large.

![Graph of an oscillatory sequence](image)

- The sequence is periodic. A set of values is repeated at periodic intervals.

![Graph of an oscillatory divergent sequence](image)

- The sequence is both oscillatory and divergent.
graph paper, and label each sketch convergent or divergent. Once they have finished have them compare their graphs and findings with their groups.

Students are now ready to find the limits using the notation \( \lim_{x \to \infty} f(n) \). Use the problems in the Infinite Sequences and Convergence Part II BLM that were convergent to show students how the table feature of their graphing calculators will give the limit numerically. Have them change the setting of the table to Indepnt: Ask and Depend: Auto and then put in increasingly large values for \( n \). They will see the values of \( f(n) \) converge towards a given value. Only after they understand what is happening, they then should be shown the algebraic methods for working the problems.

Use the algebraic method to find \( \lim_{x \to \infty} \frac{x}{x + 1} \).

Rewrite: \( \lim_{x \to \infty} \frac{x}{x + 1} \)  
\[ = \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} \]  
\[ = 1 \]  
\( \lim_{x \to \infty} \frac{1}{x} = 0 \)

Hand out the Finding Limits of Infinite Sequences BLMs. Let the students work together in their groups.

Activity 5: Using Summation Notation to Find the Sums of Series (GLE: 26)

Materials List: Working with Summation Notation BLM, graphing calculator, pencil, paper

Students need to be familiar with the following vocabulary for this activity: summation (sigma) notation, index of summation, lower limit of summation, upper limit of summation

The final activity gives students a chance to find the sums of series using summation notation. Problems include not only the familiar finite arithmetic and geometric series but also infinite geometric series that have a sum, and finite series that are neither arithmetic nor geometric.

Begin by defining sigma notation \( \sum_{i=1}^{n} a_i \), where \( a_i \) is called the summand, the numbers 1 through \( n \) are called the limits of summation and the symbol \( i \) is called the index. Any
letter can be used for the index. In expanded form, write \( \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n \).

Sigma notation can also be used to represent an infinite series \( \sum_{i=1}^{\infty} a_i \). As an example use the sum of an infinite geometric series \( \frac{a}{1-r}, \quad |r| < 1 \).

Example: Find the sum of \( \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k - 1 \)

Expanded this gives \( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \) This is an infinite geometric sequence with the sum of \( \frac{a}{1-r} \). Then \( a = 1 \) and \( r = \frac{1}{2} \) so the sum is \( \frac{1}{1-\frac{1}{2}} = 2 \)

Cover the properties of Finite Sums:

1. \( \sum_{i=1}^{n} c \cdot a_i = c \cdot \sum_{i=1}^{n} a_i \), where \( c \) is a real number
2. \( \sum_{i=1}^{n} ca_i = c \cdot \sum_{i=1}^{n} a_i \), where \( c \) is a real number
3. \( \sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \)

Example: Find the sum of \( \sum_{j=1}^{10} 4j \)

Use the properties and rewrite as \( 4 \sum_{j=1}^{10} j \)

Expand: \( 4(1 + 2 + \ldots + 10) \)

\( = 4 \left( \frac{10(1+10)}{2} \right) = 220 \)

Using the sum(seq( feature of the calculator:
Home the Working with Summation Notation BLMs. Have the students work individually on this BLM. If they use the calculator to find the sum, have them write down what they have typed in as shown above. Call on volunteers to give answers and to show their work.

Sample Assessments

General Assessments

- A writing assessment should be assigned for the unit. Students have added to their notebook glossaries throughout this unit. They have also had short writing assignments inseveral of their activities. Therefore, one of the assessments should cover this material. Look for understanding of how the term or concept is used. Use verbs such as show, describe, justify, or compare and contrast. Possible questions could be:
  Explain the difference in a recursive and in an explicit formula (nth term formula).
  Compare the end-behavior of a function with the limits of infinite sequences.

- Spiral reviews should continue with this unit. The student is asked to make connections between an arithmetic sequence and a linear function, and between a geometric sequence and an exponential function. Questions about the linear and exponential function could be mixed with use of the explicit formulas for the arithmetic and geometric sequences. There is also a connection between the end-behavior of previously studied functions and the finding of limits of infinite sequences. Giving students limit problems in which the function used is the rational function, and pointing out that they are once again looking at the end behavior of a rational function, will provide such a connection. Continue to tie the spirals to the study guide for the written test or the final exam. The Spiral BLM found in this unit, reviews the local and global behavior of functions.

- The students should work in groups to model real-life problems using both recursion and explicit formulas on problems involving sequences and series. The scoring rubric should be based on
  1. teacher observation of group interaction and work
  2. explanation of each group’s problem to class
  3. work handed in by each member of the group
Activity-Specific Assessments

- **Activity 1**: Students should demonstrate proficiency in working with the finite arithmetic and geometric sequences and series.

- **Activity 4**: Students should demonstrate proficiency in determining convergence or divergence and finding the limit of a convergent sequence.

- **Activity 5**: Students should demonstrate proficiency in working with summation notation.
Advanced Math – Pre-Calculus
Unit 8: Conic Sections and Parametric Equations

Time Frame: 2.5 weeks

Unit Description

This unit explores the characteristics of the conic sections, their graphs, and how they are interrelated in the general form. The concept of eccentricity gives a common definition to the conic sections and is used to identify the conic section and to write its equation. Methods of graphing the conic sections both by hand and by using a graphing utility are reviewed. The polar form of a conic is introduced. The unit also includes parametric equations.

Student Understandings

Students recognize a conic as the intersection of a plane and a double-napped cone. They are also able to identify the degenerate conics. Each of the conics is reviewed and applications showing how they are used in real-life situations are presented. Students are able to graph conics with and without the use of a graphing utility. Students are able to define conic sections in terms of eccentricity and to classify a conic section by looking at its equation. They are able to write and use equations of conics in polar form.

Guiding Questions

1. Can students recognize the conic as the intersection of a plane and a double-napped cone?
2. Can students write and recognize equations of the conics in standard and general forms?
3. Can students use the properties of each of the conics to solve real-life problems?
4. Can students find the eccentricities of the parabola, ellipse, and hyperbola?
5. Can students use the eccentricity to write the equations of the parabola, ellipse, and hyperbola?
6. Can students find the asymptotes of a hyperbola?
7. Can students use the asymptotes to write the equation of a hyperbola in standard form?
8. Can students write equations of conics in polar form?
9. Can students evaluate sets of parametric equations for given values of the parameter?
10. Can students graph curves that are represented by sets of parametric equations?
11. Can students use parametric equations to model motion?

Unit 8 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
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<tbody>
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<td><strong>Number and Number Relations</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Grade 9</strong></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Simplify and perform basic operations on numerical expressions involving radicals (e.g., $2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$) (N-5-H)</td>
</tr>
<tr>
<td><strong>Grade 10</strong></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Simplify and determine the value of radical expressions (N-2-H) (N-7-H)</td>
</tr>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
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<tr>
<td><strong>Grade 9</strong></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Translate between the characteristics defining a line (i.e., slope, intercepts, points) and both its equation and graph (A-2-H (G-3-H))</td>
</tr>
<tr>
<td><strong>Grade 11/12</strong></td>
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<tr>
<td>4</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>6</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)</td>
</tr>
<tr>
<td>9</td>
<td>Solve quadratic equations by factoring, completing the square, using the quadratic formula, and graphing</td>
</tr>
<tr>
<td>10</td>
<td>Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology. (A-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
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<td><strong>Grade 10</strong></td>
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<tr>
<td>3</td>
<td>Define sine, cosine, and tangent in ratio form and calculate them using technology (N-6-H)</td>
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<td>12</td>
<td>Apply the Pythagorean theorem in both abstract and real-life settings (G-2-H)</td>
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<tr>
<td><strong>Grade 11/12</strong></td>
<td></td>
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<tr>
<td>15</td>
<td>Identify conic sections, including the degenerate conics, and describe the relationship of the plane and double-napped cone that forms each conic (G-1-Ht)</td>
</tr>
<tr>
<td>16</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions(P-1-H) (P-5-H)</td>
</tr>
</tbody>
</table>
Sample Activities

In Algebra II students studied the conic sections – parabolas, circles, ellipses, and hyperbolas. Prior to the beginning of this unit, review Unit 8 to get an idea of what was covered and the vocabulary used. In the Algebra II unit, the following equations are referred to as the graphing form of the equation:

- **Circle:** \((x - h)^2 + (y - k)^2 = r^2\)
- **Parabola:** \(y = a(x - h)^2 + k\) or \(x = a(y - k)^2 + h\)
- **Ellipse:** \(\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1\)
- **Hyperbola with transverse axis horizontal:** \(\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1\)
- **Hyperbola with transverse axis vertical:** \(\frac{(y - k)^2}{b^2} - \frac{(x - h)^2}{a^2} = 1\)

This unit will refer to those equations as the standard form of the equation. Be sure students understand that they are the same. Start with Pre-test Conic Sections BLM covering what a student learned in Algebra II about conic sections to determine how much the students remember about the concepts.

**Ongoing activity: Glossary notebook**

**Materials List:** index cards 3 x 5 or 5 x 7, What Do You Know about Conic Sections and Parametric Equations? BLM, pencil, pen

Students continue the glossary activity in this unit. They will repeat the two methods used in the previous units to help them understand the vocabulary for Unit 8. Begin by having each student complete a self-assessment of his/her knowledge of the terms using a modified vocabulary self awareness chart (view literacy strategy descriptions) What Do You Know about Conic Sections and Parametric Equations? BLM. Many of the terms should be familiar because they were studied in Algebra II. Do not give the students definitions or examples at this stage. Ask the students to rate their understanding of each term with a “+” (understand well), a “?” (limited understanding or unsure) or a “–” (don’t know). Over the course of the unit students are to return to the chart, add information or examples, and re-evaluate their understanding of the terms or concepts.

Students should continue to add to their vocabulary cards (view literacy strategy descriptions) introduced in Unit 1. Make sure that the students are staying current with their cards. Time should be given at the beginning of each activity for students to bring them up to date.

Terms to add to the vocabulary list: double napped cone, conic section, locus, standard form of the equation of a circle, parabola, standard form of equation of a parabola, directrix of a parabola, focus of a parabola, ellipse, standard form of the equation of an...
ellipse, vertices, major axis, minor axis, eccentricity, hyperbola, form of the equation of a hyperbola, transverse axis, asymptotes, conjugate axis, parametric equations, plane curve, parameter

Activity 1: Working with Circles (GLEs: Grade 9: 6; Grade 10: 1, 12; Grade 11/12: 4, 9, 15, 16, 25)

Materials List: Working with Circles BLM, graph paper, paper, pencil, graphing calculator

Students need to be familiar with the following vocabulary for this activity: double-napped cone, conic section and standard form of the equation of a circle.

Begin by reviewing with the students how conic sections are formed by a plane sectioning one or both nappes of a cone at various angles. Students learned how to do this in Unit 8 of Algebra II.

Review with the students the general second-degree equation

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \]

for the conic sections, and from it write the general equation for a circle. With each of the conic sections, students should know the standard form of the equation of that conic section. For circles, it is \( (x-h)^2 + (y-k)^2 = r^2 \).

This activity gives students additional practice in writing equations of circles and working with semi-circles as functions.

Examples to use in class:

1) Write both the standard and general forms of an equation of the circle if the center of the circle is (0, 3) and a point on the circle is (4, 6).
   The distance formula gives the radius as 5.
   The standard form of the equation of the circle is \( (x-0)^2 + (y-3)^2 = 25 \)
   The general form of the equation of a circle is \( x^2 + y^2 - 6y - 16 = 0 \)

2) The equation of a semi-circle is \( y = \sqrt{8-2x-x^2} \)
Find:
   o  the center
   o  the radius
   o  the domain and range
   o  sketch the graph
(a) Write the equation in standard form for a circle:

\[ y = \sqrt{9 - (x + 1)^2} \]
\[ y^2 = 9 - (x + 1)^2 \]
\[ (x + 1)^2 + y^2 = 9 \]

*The center is (-1, 0), the radius is 3, the domain is \{x: -4 ≤ x ≤ 2\} and the range is \{y: 0 ≤ y ≤ 3\}*.

Have the students sketch the graph by hand, then verify the results using the graphing calculator.

![Graph of a circle](image)

Hand out the Working with Circles BLMs for the students to work in class. Be sure that they show the work needed to find the answers. Graphs should be done by hand and verified using the calculator.

**Activity 2: Parabolas as Conic Sections (GLEs: Grade 9: 13; Grade 11/12: 4, 9, 15, 16, 25)**

Materials List: Parabolas as Conic Sections BLM, graphing calculator, pencil, graph paper

Students need to be familiar with the following vocabulary for this activity: parabola, standard form of equation of a parabola, directrix of a parabola, focus of a parabola

Students have studied parabolas as a polynomial function. This activity looks at a parabola as a conic section. Review with the students the definition learned in Algebra II.

- A parabola is the set of all points in a plane that are equidistant from a fixed point called the focus and a fixed line called the directrix. The midpoint between the focus and directrix is the vertex of the parabola. Parabolas can be both vertical and horizontal.

The standard form of the equation of a parabola with vertices at (0,0) and the focus lying \(d\) units, \(d > 0\), from the vertex is given by

- \(x^2 = 4dy\) if the axis is vertical and
- \(y^2 = 4dx\) if the axis is horizontal.

The graphs of the parabola would open up or to the right. If \(d < 0\) the orientation of the parabola would change and the graph would open down or to the left.
If the vertex of the parabola is at the point \((h, k)\), then the standard forms of the equation are

- \((x - h)^2 = 4d(y - k)\) if the axis is vertical and
- \((y - k)^2 = 4dx(x - h)\) if the axis is horizontal.

In this activity, students will need to be able to identify the vertex, focus, and directrix of the parabola given its equation, write the equation given the identifying features, and look at a series of functions that are “half parabolas”.

Example 1: Given the equation \(x^2 = 8y\). Find the vertex, focus, and directrix of the parabola.

*The vertex of this parabola is \((0, 0)\) and the axis of the parabola is vertical. The value of \(d\) is 2. Therefore, the focus is \((0, 2)\) and the directrix is \(y = -2\).*

Example 2: Determine the vertex, focus, and directrix of the parabola given by the equation \(y^2 - 2y - 4x = -1\).

*Begin by writing the equation in standard form: \((y - 1)^2 = 4x\). The vertex is at \((0, 1)\) and the axis of the parabola is horizontal. \(d = 1\) so the focus is the point \((1, 1)\) and the directrix is the line \(x = -1\).*

Example 3: Find the standard form of the equation of the parabola with vertex \((5, 1)\) and focus \((2, 1)\).

*The vertex and focus lie on the line \(y = 1\) so the axis of the parabola is horizontal. The graph of the parabola will open to the left since \(d = -3\). The equation in standard form is \((y - 1)^2 = 4(-3)(x - 5)\), \((y - 1)^2 = -12(x - 5)\).*

Hand out the Parabolas as Conic Sections BLMs for the students to work in class. Be sure that they graph by hand using the graph paper.

**Activity 3: Eccentricity (GLEs: Grade 9: 6, 13, Grade 10: 1, Grade 11/12: 4, 9, 15, 16)**

**Materials List:** Using Eccentricity to Write Equations and Graph Conics BLM, graph paper, graphing calculator, paper, pencil

Students need to be familiar with the following vocabulary for this activity: locus, eccentricity, ellipse, standard form of the equation of an ellipse, vertices, major axis, minor axis, foci of the ellipse, hyperbola, form of the equation of a hyperbola, transverse axis, asymptotes, conjugate axis, foci of the hyperbola.
Review with the students the terms associated with the ellipse and hyperbola.

Example 1:

Sketch and label the following ellipse: \(9x^2 + y^2 = 36\).

1) Rewrite in standard form: \(\frac{x^2}{4} + \frac{y^2}{36} = 1\).
2) The major axis is vertical, \(a = 6\), and the vertices are (0, 6) and (0, -6).
3) The minor axis is horizontal, \(b = 2\), and the endpoints of this axis are at (2, 0) and (-2, 0).
4) Graph using the calculator, \(Y1: \sqrt{36 - 9x^2}\) or sketch on graph paper.

5) Graph:

6) The foci of the ellipse are \(c\) units from the center on the major axis. For this problem,
\[ c = \sqrt{b^2 - a^2} = \sqrt{36 - 4} = 4\sqrt{2} \]
7) foci are (0,4\(\sqrt{2}\)) and (0,-4\(\sqrt{2}\)).

Example 2:

Sketch and label the following hyperbola: \(x^2 - 4y^2 = 16\).

1) Rewrite in standard form: \(\frac{x^2}{16} - \frac{y^2}{4} = 1\).
2) The transverse axis is horizontal so the vertices occur at (4, 0) and (-4, 0).
3) The ends of the conjugate axis are found at (0, 2) and (0, -2).
4) Using those 4 points sketch a rectangle. The diagonals of that rectangle are the asymptotes of the hyperbola. Their equations are \(y = \frac{1}{2}x\) and \(y = -\frac{1}{2}x\).
5) To graph use the graphing calculator: Y1: \((\sqrt{x^2 - 16}) + 2\)  
Y2: \((-\sqrt{x^2 - 16}) + 2\) with the asymptotes \(y = \frac{1}{2}x\) and \(y = -\frac{1}{2}x\).

6) Graph:

7) The foci are \(c\) units from the center on the transverse axis and \(c = \sqrt{a^2 + b^2}\). The foci are found at \((4\sqrt{17}, 0)\) and \((-4\sqrt{17})\).

In Activity 2 students learned how to define a parabola using the focus (a fixed point) and the directrix (a fixed line). This same point and line will be used to define the ellipse and hyperbola.

Begin with a fixed line, \(d\), and a fixed point \(F\). The conic is the set of all points, \(P\), on the plane where the ratio of its distance from the point \(F\) (\(PF\)) and the distance from the line \(d\) (\(PD\)) is a constant. Call that constant \(e\) the eccentricity of the conic. This gives the following: \(\frac{FP}{PD} = e\)

With the parabola \(e = 1\); that is, the distance from the point to the directrix and the point to the focus was equal. In the graphic below note that \(FP_1 = FP_2 = FP_3 = FP_4 = 1\)
Conics are changed by changing the value of the eccentricity. If $0 < e < 1$, the conic is an ellipse. In the graphic below, $e = \frac{1}{2}$.

Finally if $e > 1$, then the conic is a hyperbola. In the graphic below $e = 2$.

Example 3:
Find the equation of a conic whose eccentricity is $\frac{4}{5}$, a focus at the point $(1, -2)$ and directrix is the horizontal line $y = \frac{17}{4}$.

Solution:
Use the distance formula to find $PF$ and $PD$ given the point $P(x, y)$, the focus $F(1, -2)$, and point $D\left(x, -\frac{17}{4}\right)$ where $D$ is the foot of the perpendicular from $P$ to the directrix.

$y = -\frac{17}{4}$

$\frac{PF}{PD} = \frac{4}{5}$

$PF = \frac{4}{5} PD$

$\sqrt{(x-1)^2 + (y-2)^2} = \frac{4}{5}\left(y + \frac{17}{4}\right)$
Squaring both sides and simplifying:
\[ x^2 - 2x + 1 + y^2 + 4y + 4 = \frac{16}{25}\left(y^2 + \frac{17}{2}y + \frac{289}{16}\right) \]
\[ 25x^2 - 50x + 25y^2 + 100y + 125 = 16y^2 + 136y + 289 \]
\[ 25x^2 + 9y^2 - 50x - 36y = 164 \]

The eccentricity for ellipses and hyperbolas can also be defined as \( e = \frac{c}{a} \) where \( c \) is the distance from the center to the focus and \( a \) is the distance from the center to the vertex.

Example 4:

An ellipse has foci at (5, 1) and (-1, 1). The length of its major axis is 8. Find its eccentricity and equation in standard form.

Solution:

The major axis lies on the line \( y = 1 \).
The center of the ellipse is the midpoint of the foci. It is the point (2, 1).
The distance \( c \) from the center to each focus is 3.
The length of the major axis is 2a. Therefore \( a = 4 \).

\[ e = \frac{c}{a} = \frac{3}{4} \]

The equation of an ellipse in standard form is \( \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \) and \( b^2 = a^2 - c^2 \). Therefore \( b^2 = 7 \).

The equation is \( \frac{(x-2)^2}{16} + \frac{(y-1)^2}{7} = 1 \)

In this activity, both definitions of eccentricity are used. The activity begins with identification of the conic from the given eccentricity. Students then use the information given to find the equation and graph the conic. Hand out the Using Eccentricity to Write Equations and Graph Conics BLMs. Let students work in their groups on the problems.

**Activity 4: Polar Equations of Conics (GLEs: Grade 10: 3; Grade 11/12: 15, 16, 25)**

Materials List: Polar Equations of Conics BLM, graphing calculator, pencil, paper

There is an alternate way to define the conics using polar coordinates. Rectangular coordinates place the most importance on the location of the center of the conic, while polar coordinates place more importance on where the focus of the conic is located. In a
polar equation for a conic, the pole is the focus of the conic, and the polar axis lies along the positive x-axis. Eccentricity $e$ of the conic identifies the type of conic, and the definition given in Activity 3, again applies here.

- Begin with a fixed line, $d$, and a fixed point $F$. The conic is the set of all points, $P$, on the plane where the ratio of its distance from the point $F$ ($PF$) and the distance from the line $d$ ($PD$) is a constant. (See the figure on page 11.) $D$ is the point on the directrix closest to the point $P$. Call that constant $e$, the eccentricity of the conic. This gives the following: \[ \frac{FP}{PD} = e \]

Let $p$ be the distance between the focus (pole) and the directrix of a given conic. Then, the polar equation for a conic takes one of the following two forms:

\[ r = \frac{ep}{1 \pm \sin \theta} \quad \text{or} \quad r = \frac{ep}{1 \pm \cos \theta} \]

The equation that uses $1 \pm \sin \theta$ has a horizontal line as directrix. The equation that uses $1 \pm \cos \theta$ has a vertical line as directrix. In the diagram below one of the foci is at the origin. The directrix $x = k$ is perpendicular to the positive x axis. The length $FB$ is $r \cos \theta$. The line segment $PD = k - r \cos \theta$.

The equation is derived as follows:

Using the definition of eccentricity \[ \frac{FP}{PD} = e \] and rewriting $FP = ePD$

1) $FP = ePD$, substituting $r$ for $FP$ and $k - r \cos \theta$ for $PD$
2) $r = e(k - r \cos \theta)$
3) $r = ek - er \cos \theta$, solving for $r$ the equation of the conic section in polar form is obtained
4) \[ r = \frac{ek}{1 + e \cos \theta} \], where \( k \) is the distance \( p \) between the focus (pole) and the directrix

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
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<tbody>
<tr>
<td>When ( x = k ),</td>
<td>Directrix is perpendicular to the polar axis at a distance ( k ) units to the right of the pole.</td>
</tr>
<tr>
<td>( r = \frac{ek}{1 + e \cos \theta} )</td>
<td></td>
</tr>
<tr>
<td>When ( x = -k ),</td>
<td>Directrix is perpendicular to the polar axis at a distance ( k ) units to the left of the pole.</td>
</tr>
<tr>
<td>( r = \frac{ek}{1 - e \cos \theta} )</td>
<td></td>
</tr>
<tr>
<td>When ( y = k ),</td>
<td>Directrix is parallel to the polar axis at a distance ( k ) units above the pole.</td>
</tr>
<tr>
<td>( r = \frac{ek}{1 + e \sin \theta} )</td>
<td></td>
</tr>
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</tr>
<tr>
<td>( r = \frac{ek}{1 - e \sin \theta} )</td>
<td></td>
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</tbody>
</table>

The equation for the directrix is \( r \cos \theta = \pm k \) if the directrix is a vertical line, and \( r \sin \theta = \pm k \) if the line is horizontal.

Example 1: Given the eccentricity, \( e = \frac{1}{2} \) and the directrix \( x = -1.5 \), find the polar equation for this conic section. Verify the answer by graphing the polar conic and directrix on the same screen using the graphing calculator.

**Solution:**

Using \( r = \frac{ek}{1 - e \cos \theta} \) with \( e = \frac{1}{2} \) and \( k = -1.5 \)

\[
\frac{r}{1 - 5 \cos \theta} = \frac{5(1.5)}{1 - 5 \cos \theta} = \frac{7.5}{1 - 5 \cos \theta}
\]

The directrix is \( r = \frac{-1.5}{\cos \theta} \)

The graph is shown below. Be sure that the window is squared.
Example 2:
Discuss and graph the equation \( r = \frac{8}{2 - \sin \theta} \). Identify the conic. Find the eccentricity.
Write the equation for the directrix. Graph the conic and its directrix.

Solution:
Rewrite \( \frac{8}{2 - \sin \theta} \) in the form \( \frac{ek}{1 - e\sin \theta} \) by multiplying every term by \( \frac{1}{2} \). Then \( r = \frac{4}{1 - \frac{1}{2} \sin \theta} \). The eccentricity is \( \frac{1}{2} \) so the conic is an ellipse. One focus is at the pole and the directrix is parallel to the polar axis a distance of 8 units below the polar axis. The major axis is along the pole. The equation for the directrix is \( y = -8 \) which is \( r = \frac{-8}{\sin \theta} \) in polar form.

Square the window using the following settings: \( X_{\text{min}} = -13.63, X_{\text{max}} = 13.63, Y_{\text{min}} = -9, Y_{\text{max}} = 9 \).

The graph:

Hand out the Polar Equations of Conics BLMs. Let students work in their groups to complete the activity.

Activity 5: Sketching a Plane Curve Given by Parametric Equations (GLEs: 4, 6, 10)

Materials List: Plane Curves and Parametric Equations BLM, graphing calculator, graph paper, pencil

Students need to be familiar with the following vocabulary for this activity: parametric equations, plane curve, parameter.

Parametric equations differ from rectangular equations in that they are useful in finding both the time and the positions of an object. By plotting the points in the order of increasing values of \( t \), one traces the curve in a specific direction. This is called the
orientation of the curve. Using this, it is also possible to see if two curves (1) intersect at different times or (2) reach the same point at the same time.

- Present the definition: Let \( x = f(t) \) and \( y = g(t) \), where \( f \) and \( g \) are two functions whose common domain is some interval \( I \). The collection of points defined by \((x, y) = (f(t), g(t))\) is called a plane curve. The equations \( x = f(t) \) and \( y = g(t) \) where \( t \) is in \( I \), are called parametric equations of the curve. The variable \( t \) is called a parameter.

Example:

Discuss the plane curve defined by the parametric equations \( x = 2t^2 \) and \( y = 3t \) with the interval \(-1 \leq t \leq 1\)

For each value of \( t \) in the interval \([-1, 1]\), there corresponds a value \( x \) and a value \( y \).

Set up a table such as the one below:

Graph the parametric equations using a graphing calculator.

1: Set the mode to PARametric and choose either Connected or Dot
2: Press Y=
3. Enter \( X1T = 2T^2 \)
   \( Y1T = 3T \)
4. Set the window as shown below:

![Window Settings]

5. GRAPH

![Graph Image]
6. Notice the direction the graph is drawn. This direction shows the orientation of
the curve. Using TRACE will also produce the orientation as shown below.

Finally, find the corresponding rectangular equation by eliminating the parameter \( t \) from
the parametric equations \( x = 2t^2 \) and \( y = 3t \).

\[ y = 3t \text{ then } t = \frac{y}{3} \]

\[ x = 2t^2 \]

\[ = 2 \left(\frac{y}{3}\right)^2 \]

\[ x = \frac{2y^2}{3} \]

This is the equation of a parabola with vertex at \((0, 0)\).

Example 2:

Discuss the plane curve defined by the parametric equations \( x = t - 3 \), \( y = 2t + 4 \) with the
interval \(-1 \leq t \leq 2\).

a. Complete the table.
b. Plot the points \((x, y)\) from the table labeling each point with the parameter
value \( t \).
c. Describe the orientation of the curve.
d. Find the corresponding rectangular equation by eliminating the parameter \( t \).

(a) *Set up a table*

<table>
<thead>
<tr>
<th>( t )</th>
<th>-1</th>
<th>- .75</th>
<th>-.25</th>
<th>0</th>
<th>.25</th>
<th>.50</th>
<th>.75</th>
<th>1</th>
<th>1.25</th>
<th>1.50</th>
<th>1.75</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>-2</td>
<td>-2.25</td>
<td>-1.5</td>
<td>-1</td>
<td>- .75</td>
<td>-.5</td>
<td>-.25</td>
<td>0</td>
<td>.25</td>
<td>.50</td>
<td>.75</td>
<td>1</td>
</tr>
<tr>
<td>( y )</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>5.5</td>
<td>6.0</td>
<td>6.5</td>
<td>7.0</td>
<td>7.5</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>
(b) Have students graph using graph paper. Plot each point and then lightly trace. Show the orientation by placing arrows showing the movement as \( t \) increases. Students should have a line segment with endpoints (-4, 2) and (-1, 8). The movement will be from left to right.

(c) The corresponding equation is \( y = 2x + 10 \).

It is also possible to look at the velocity of an object using parametric equations. Remind the students that velocity shows both speed and direction. In the problem below, when the particle moves right the velocity is positive. When the particle moves left, it is negative. It is also possible to see the particle speed up and slow down. By using TRACE students can see the movement and record the times that the particle turns. Have the students use the following window:

```
WINDOW
Tmin=0
Tmax=10
Tstep=.1
Xmin=0
Ymin=0
Xmax=100
Xscale=1
Yscale=1
```

Present the following problem to the students:

An object moves to the right in such a way that its displacement from the vertical axis is \( x = 3t^3 - 30t^2 + 64t + 57 \), for \( t \geq 0 \). With the graphing calculator in parametric mode, plot the path of the object using TRACE. Use the right hand arrow throughout the TRACE process. Mark the times the particle speeds up or slows down. Write a paragraph describing the motion. Include the approximate times and places where the object reverses direction, and the time intervals during which the object is traveling to the right and to the left. Describe the speed.

A writing of this type is a math learning log. Students in this course have been writing math learning logs as part of their general assessments in each of the units. A math learning log is a form of a learning log (view literacy strategy descriptions) A learning log is a notebook that students keep in math classrooms in order to record ideas, questions, reactions and new understandings. This forces students to “put into words” what they have learned from this problem. This process offers a reflection of understanding that can lead to further study and alternative learning paths. It combines writing and reading with content learning. Throughout this course students have been doing this type of writing as part of their written assessments.

Have students share their responses with a partner for feedback and clarification. Once this is done have student volunteers share their answers with the class. Conduct a class discussion of the accuracy of their answers.
Solution: The object starts out at \( x = 57 \) miles when \( t = 0 \). The velocity is positive since the particle is moving to the right. The particle slows down as it gets close to 1.3 hours. There it stops, where \( x \) is about 96 miles. It turns and starts moving to the left slowly at first, then faster and faster. As the particle gets close to 5.3 hours it slows, then stops at about .1 mile. It turns again and begins moving to the right. The speed continues to increase throughout the remainder of the problem.

Once students finish the writing activity hand out the Plane Curves and Parametric Equations BLMs. Let the students complete this activity in their groups. Instructions for the worksheet are as follows:

- Part I. Fill in the table and sketch by hand the curve given by the parametric equations. Describe the orientation of the curve.
- Part II. Use technology to graph the curve. Eliminate the parameter. Students should note that eliminating the parameter and rewriting the equations in rectangular form can change the ranges of \( x \) and \( y \). In such cases \( x \) and \( y \) should be restricted so that its graph matches the graph of the parametric equations.
- Part III In the rectangular coordinate system the intersection of two curves can be found either graphically or algebraically. With parametric equations, it is possible to distinguish between an intersection point (the values of \( t \) at that point are different for the two curves) and a collision point (the values of \( t \) are the same).

Activity 6: Group Problem Solving: Modeling Motion Using Parametric Equations (GLE: 10)

Materials List: Modeling Motion Using Parametric Equations BLM, graphing calculator, graph paper, paper, pencil

Students have learned how to solve problems dealing with the height of a projectile at time \( t \) in earlier units. Parametric equations allow a description of both the horizontal and vertical path of a projectile at time \( t \). If a projectile is subject to no other force than gravity then
- \( x(t) \) = horizontal position of the projectile at time \( t \)
- \( y(t) \) = vertical position of the projectile at time \( t \)

The path of the projectile is described by the equations
- \( x = (v_o \cos \theta) t \)
- \( y = -\frac{1}{2} gt^2 + (v_o \sin \theta)t + h \)

where \( v_o \) is the initial velocity, \( \theta \) is the angle at launch, \( t \) is time and \( g \) is the acceleration due to gravity (32 \( \text{ft/sec}^2 \) or 9.8\( \text{m/sec}^2 \)).

Example: A rocket is launched from ground level with an initial velocity \( v_o = 1200 \text{ mph} \) at an angle of inclination \( \theta = 32^\circ \). Find:
a) Find the parametric equations that describe the position of the rocket as a function of time.
b) When is the rocket at its maximum height? What is the maximum height?
c) How long does it take to return to earth?
d) What is the distance it travels horizontally from the launch site to return to earth?

Solution:

First convert the initial velocity to ft/sec.

\[
1200 \text{ miles} \cdot \frac{1200 \text{ feet}}{3600 \text{ sec}} = 1760 \text{ feet/sec}
\]

\[
a) \ x(t) = 1760(\cos 32^\circ)t \text{ and } y(t) = -16t^2 + (1760\sin 32^\circ)t + 0
x(t) \approx 1492.5t \text{ feet and } y(t) \approx 932.8t - 16t^2 \text{ feet}
\]

\[
b) \ The \ parametric \ equations \ form \ a \ parabola \ so \ to \ find \ the \ height \ and \ the \ time \ it \ takes \ to \ get \ there \ use \ \left(-\frac{b}{2a}\right)
\]

\[
t = \frac{-932.8}{2(-16)} = 29.15 \text{ seconds}
\]

\[
y(29.15) = 932.8(29.15 - 16(29.15)^2)
\]

\[= 13,595.6 \text{ feet} \approx 2.575 \text{ miles}
\]

\[c) \ \text{Launch time is represented by } t = 0, \text{ solve } 0 = 932.8t - 16t^2 \text{ to find the time the rocket returns to the ground. } t = 58.3 \text{ seconds}
\]

\[d) \ The \ horizontal \ position \ of \ the \ rocket \ at \ the \ time \ it \ strikes \ the \ earth \ is \ x(58.3) = (1492.5)(58.3) = 87,012.75 \text{ feet} \approx 16.48 \text{ miles.}
\]

Hand out the Modeling Motion Using Parametric Equations BLMs. Allow the students to work in their groups to solve the problems.

Sample Assessments

General Assessments

- A writing assessment should be assigned for the unit. A good one for this unit is to have the students explain how all of the conic sections can be obtained by intersecting a double napped cone and a plane.
- Spiral reviews continue with this unit. Students are called on to use previously learned algebra to find the general equations and the equations in standard form of each of the conics. They will need to update their skills in this area. A spiral review of this material would be helpful. They are also called upon to work again with domain, range, and zeros of functions. This material would also make a
good spiral. Finally they have to once again work with polar coordinates and rectangular and polar equations. The General Assessments Spiral BLM is on this topic. Assign it along with a review prior to Activity 4.

- A good activity for this unit is a group competition. Make out a set of questions that covers all of the material in the unit, then divide that set so that each group gets one portion. Each group will have a certain number of minutes to work on the problems, then they must pass them on to the next group. At the end of the time period, all members of each group will hand in their answer sheets plus work done on the problems. The number of questions will depend on the number of groups and the time you have for the activity. The scoring rubric should be based on teacher observation of group interaction and work, as well as, the number of correct answers.

Activity-Specific Assessments

- **Activities 1 and 2:** Students should demonstrate proficiency working with functions that result from conic sections.

- **Activity 3:** Students should demonstrate proficiency in identifying various conic sections through the given eccentricity or through the general equation.

- **Activity 5:** Students should demonstrate proficiency in sketching curves of parametric equations.